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Wave-function transformations by general SU(1, 1) single-mode squeezing and analogy to Fresnel transformations in wave optics

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Abstract

Following Dirac's assertion: "… for a quantum dynamic system that has a classical analogue, unitary transformation in the quantum theory is the analogue of contact transformation in the classical theory", we find that the general SU(1, 1) single-mode squeezing operator F just corresponds to the generalized Fresnel transform (GFT) in wave optics. We derive the normal product form and canonical coherent state representation of F, whose matrix element in the coordinate representation is just the GFT. It is shown that F is a faithful representation of symplectic group which indicates that two successive GFTs is still a GFT. Applications of F in some other optical transforms, such as the Fresnel-wavelet transform, are presented.

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1. Introduction

When we read the paragraph of Dirac's famous book *Principles of Quantum Mechanics* [1]: "… for a quantum dynamic system that has a classical analogue, unitary transformation in the quantum theory is the analogue of contact transformation in the classical theory", we are naturally challenged by such an interesting question: Corresponding to the optical Fresnel transform in wave optics, what is its genuine quantum mechanical analogy? Such a question, so far as our knowledge is concerned, has not been posed

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in the literature before. So we are motivated to find a unitary operator F to correspond to a generalized Fresnel transform (GFT), which turns out to be a general SU(1, 1) single-mode squeezing operator, as we shall demonstrate this in the following. In the last era, GFT has been widely implemented in Fourier optics: optical imaging, optical propagation, optical engineering and optical instrument design, therefore to study it in the context of quantum optics is really worthwhile. Note the fact that two successive GFTs is still a GFT, the operator F we propose must have similar well-behaved properties as possessed by successive GFTs. The work is arranged as follows: In Section 2 we derive the normal ordered form of F and also its coherent representation, from which we prove the important symplectic group [2] multiplication rule of F. Section 3 is devoted to discussing the applications of this unitary operator, i.e., introducing some new optical transformations by combining Fresnel transform and wavelet transform.

2. The general SU(1, 1) single-mode squeezing operator

The way we search for the candidate operator F is employing the coherent state method to link GFT to F. The GFT of a function $f(x_1)$ is defined as [3–5]

$$g(x_2) = \int_{-\infty}^{\infty} \mathscr{K}(A,B,C,D;x_2,x_1) f(x_1)\,\mathrm{d}x_1, \tag{1}$$

where

$$\mathscr{K}(A,B,C,D;x_2,x_1) = \frac{1}{\sqrt{2\pi \mathrm{i}B}} \exp\left[\frac{\mathrm{i}}{2B}(Ax_1^2 - 2x_2x_1 + Dx_2^2)\right], \tag{2}$$

is the transform kernel with the real parameters A, B, C and D being restricted by $AD - BC = 1$. It is easily seen that when $\mathscr{K}$ is $\exp(\mathrm{i}x_2x_1)$, the GFT reduces to the well-known Fourier transform, which is adopted to express mathematically the Fraunhofer diffraction; and if $\mathscr{K}$ is $\exp[\mathrm{i}(x_2 - x_1)^2]$, the GFT then describes a Fresnel diffraction. Recently, much attention has been paid to fractional Fourier transform (FRFT) [6], which can be implemented in the light propagation in gradient-index media [7,8], or in some lens systems [9,10], etc. By setting $A = D = \cos\theta$ and $B = -C = \sin\theta$, we see that FRFT is also a special case of the GFT.

To our knowledge, the quantum mechanical unitary operator which takes the optical Fresnel transform as its classical counterpart has not been reported in the literature before. We find that the concrete form of $F(A,B,C,D)$ should be

$$F = \exp[\mu a^{\dagger 2}] \exp\left[\left(a^\dagger a + \frac{1}{2}\right)\ln\tau\right] \exp[\nu a^2], \tag{3}$$

where

$$\mu \equiv -\frac{(A-D) + i(B+C)}{2[A+D+i(B-C)]},$$
$$\nu \equiv \frac{(A-D) - i(B+C)}{2[A+D+i(B-C)]},$$
$$\tau \equiv \frac{2}{A+D+i(B-C)}, \tag{4}$$

where a and $a^\dagger$ are the Bose operators, $[a, a^\dagger] = 1$. From (3) one can see that F is a general SU(1, 1) single-mode squeezing operator (which is more complicated than the usual squeezing operator with a complex squeezing parameter) in a disentangled form, because $a^{\dagger 2}$, a^2 and $a^\dagger a + \frac{1}{2}$ are three generators of SU(1, 1) Lie algebra. For squeezed states and squeezing operators we refer to [13,14]. We shall show that the GFT kernel $\mathscr{K}$ in (2) is just the transformation matrix element of GFO in the coordinate representation $|x\rangle$, i.e.