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Reprint of : Dynamics of coupled vibration modes in a quantum non-linear mechanical resonator

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H I G H L I G H T S

- If two flexural modes are mechanically coupled and one of them is driven the other mode responds.
- The response is similar to the parametric resonance in the classical physics.
- In the quantum case, the non-driven mode responds even below the classical threshold.
- The magnitude of the response is comparable to the amplitude of the zero-point motion.

A R T I C L E I N F O

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We investigate the behaviour of two non-linearly coupled flexural modes of a doubly clamped suspended beam (nanomechanical resonator). One of the modes is externally driven. We demonstrate that classically, the behavior of the non-driven mode is reminiscent of that of a parametrically driven linear oscillator: it exhibits a threshold behavior, with the amplitude of this mode below the threshold being exactly zero. Quantum-mechanically, we were able to access the dynamics of this mode below the classical parametric threshold. We show that whereas the mean displacement of this mode is still zero, the mean squared displacement is finite and at the threshold corresponds to the occupation number of 1/2. This finite displacement of the non-driven mode can serve as an experimentally verifiable quantum signature of quantum motion.

1. Introduction

Observation of quantum effects in mechanical resonators, first reported in Ref. [1] for a GHz resonator read out by a superconducting qubit, became a breakthrough in the field of nano- and optomechanics. Subsequently, quantum effects were also confirmed in a mechanical drum resonator coupled to a superconducting microwave cavity [2] and in cavity optomechanical systems [3,4]. This breakthrough shifted the interest to the possible use of mechanical systems as quantum state transducers [5–7] and eventually to the construction of integrated coherent mechanical-based circuits. Investigation of fundamental properties of coupled mechanical resonators is essential to achieve this goal.

Coupling of linear mechanical resonators or different modes of the same resonator has been extensively studied in the literature [8–12]. Recently, first experimental [13–15] and theoretical [16–19] studies of non-linearly coupled resonators were made available. They are facilitated by the fact that many available nanomechanical systems, such as suspended beams or membranes, are inherently non-linear due to elongated-induced stress. In the single-electron tunneling regime, non-linearities may be even stronger due to the Coulomb effects and can be controlled by nearby electric gates [20–23]. A basic property of a non-linear mechanical system is interaction between different vibrational modes. Stronger nonlinearity induces stronger coupling between these modes, which is highly beneficial for building integrated coherent circuits, classical as well as quantum ones.

Classically, non-linear systems exhibit extremely rich dynamical behavior, and in seemingly close situations they may behave very differently. Quantum effects in non-linear systems have been discussed in several contexts, including mechanical resonators [24–26], and are generally recognized as a very complex and difficult problem. Non-linearity is essential for quantum position detection, since the mean expectation value of the displacement operator in a linear system is zero. The non-linear nature of a mechanical resonator can facilitate the transition into the quantum regime [27].

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In this paper, we concentrate on one important aspect of quantum non-linear mechanical resonators, which is interaction between vibration modes. We specifically consider a situation, when only one mode is externally driven. We first solve the classical problem and find that it is reminiscent of the parametrically driven oscillator, so that the non-driven mode only gets excited if the driving force exceeds certain threshold. Below this threshold, the classical displacement of the non-driven mode is exactly zero. Subsequently, we solve the quantum problem below this (classical) threshold using the Lindblad master equation technique and discover that quantum-mechanically, the non-driven mode gets excited to the states with non-zero number of phonons, up to the average occupation of one-half. This means that the occupation of the non-driven mode below the threshold is a quantum-mechanical effect and can serve as a signature of quantum motion. It also opens the way for detailed investigation of quantum dynamics of coupled mechanical oscillators such as for example entanglement generation or quantum state transfer between the modes.

The paper is organized as follows. In Section 2 we outline the model and derive equations of motion. We continue with the classical treatment of the equations in Section 3, where we calculate the amplitudes of mechanical motion of the two modes. We quantize the system and investigate its quantum dynamics, solving the Lindblad equations, in Section 4. In Section 5 we present the conclusions.

2. Model

To describe the interaction between the flexural modes of a doubly clamped nanomechanical beam, shown in Fig. 1, we use the Euler–Bernoulli equation [28]. We first derive the Hamiltonian of the beam. The latter is subject to the driving force $\tilde{F}(\tilde{t})$, which can be of optical or magnetomotive in origin and induces the time-dependent bending profile $\tilde{u}(\tilde{y}, \tilde{t})$. Displacement of the beam results in elongation which in turn induces the non-linear tension $\tilde{T}$. For simplicity, we use below the reduced coordinate $y = \tilde{y}/L$ along the beam and the reduced displacement $u = \tilde{u}/r$, where L and r are the length and the radius of the beam, which we take to be of a circular cross-section. Also we introduce the dimensionless time $t = \sqrt{D/\rho S L^4}\,\tilde{t}$, with D being the bending rigidity, ρ the mass density and S the area of the cross section of the beam. The dimensionless tension $T = L^2\tilde{T}/D$ is given by Ref. [28],

$$T = T_0 + \frac{K}{2}\int_0^1 dy (u''(y, t))^2, \tag{1}$$

where T_0 is the residual tension of the beam and $K = r^2 S/I$, with I being the second moment of inertia. We denote by primes and dots spatial and temporal derivatives, respectively. The applied

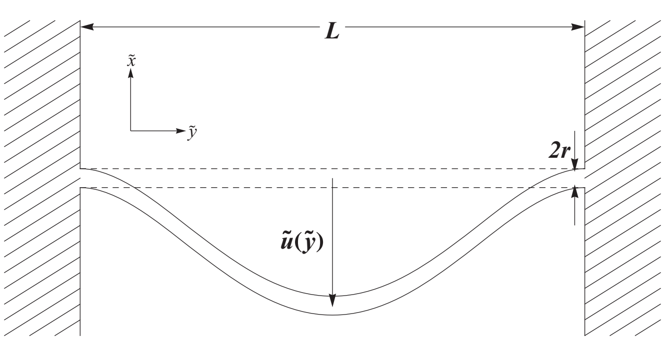

Fig. 1. Schematic representation of a doubly clamped nanomechanical resonator of the length L and the radius r. An applied force induces the bending profile $\tilde{u}(\tilde{y})$ as indicated.

force $\tilde{F}$, for which we use the dimensionless expression $F = L^4\tilde{F}/Dr$, can have static F_{dc} and time dependent F_{ac} components which result in dc and ac displacements of the beam $u(y, t) = u_{dc}(y) + u_{ac}(y, t)$ respectively. The equations of motion for these components have the form [29,32,30]

$$u''''_{dc} - T_{dc}u''_{dc} = F_{dc}; \tag{2}$$

$$\ddot{u}_{ac} + \eta\dot{u}_{ac} + \mathcal{L}[u_{ac}] - (T^* u''_{dc} + T_{ac}u''_{ac}) - T^* u''_{ac} = F_{ac}. \tag{3}$$

Here, T_{dc} is the sum of the residual tension and the one resulting from the dc displacement; T_{ac} is the tension term which contains all terms that are linear in u_{ac}, and T^* is quadratic in u_{ac}. The operator $\mathcal{L}[u]$ is defined as

$$\mathcal{L}[u] = u'''' - T_{dc}u'' - T_{ac}[u]u''_{dc}. \tag{4}$$

The first three terms on the left-hand side of Eq. (3) determine the linear response of the system. The last two terms introduce the nonlinearities with u_{ac}^2 and u_{ac}^3.

The eigenfunctions $\xi_n(y)$ and the eigenvalues ω_n of the operator $\mathcal{L}$ correspond to the mode shapes and frequencies of these modes respectively [30]. The ac displacement is expanded in terms of the mode shapes as $u_{ac}(y, t) = \sum_{n=1}^{\infty}\xi_n(y)u_n(t)/\sqrt{2\omega_n}$. Inserting this expansion in Eq. (3) and taking the driving force to be a periodic function with the amplitude F_0 and the frequency ω_d provides a set of coupled equations of motion for the displacements of the modes, $u_n(t)$,

$$\ddot{u}_n + \eta_n\dot{u}_n + \omega_n^2 u_n + \omega_n K\sum_{i,j}(2A_i I_{nj} + A_n I_{ij})u_i u_j + \omega_n K\sum_{ijk} I_{ij}I_{kn}u_i u_j u_k$$
$$= 2\omega_n S_n F_0 \cos(\omega_d t), \tag{5}$$

where the summation runs over all the eigenmodes, the values of $I_{ij} = \int\xi_i'(y)\xi_j'(y)\,dy$ depend only on the shapes of the modes i and j, $S_n = \int\xi_n(y)\,dy$ is the mean displacement of the mode n per unit deflection, and $A_i = \int u_{dc}'(y)\xi_i'(y)\,dy$ depend on the static displacement. These coefficients can be calculated numerically. In the case of zero dc displacement, A_i is zero, and the last term on the left-hand side of the Eq. (5) couples the modes. Here we assume that the beam is in the strong bending regime where the dc displacement is big enough so that the geometrical nonlinearity plays an important role, but the time-dependent component of the deflection is small enough, and one can disregard nonlinearities which it causes. In this situation, we can disregard the last term in Eq. (5). This statement imposes constraints on the ac displacement which can be found from the following inequality, $\sum_{i,j}(2A_i I_{nj} + A_n I_{ij})u_i u_j \gg \sum_{ijk} I_{ij}I_{kn}u_i u_j u_k$, see Ref. [17] for more details.

Note that since the external force is spatially homogeneous, it only can excite modes with odd n, for which $S_n \neq 0$. Our focus here is mode interaction, and therefore we only consider two modes, one of which is odd (driven), and another is even (not driven). Specifically, we take n=2 and n=3. This choice has an additional convenience since, as we show below, these modes are coupled the strongest in the quantum regime due to the frequency matching. For simplicity, we also disregard the terms with i=j in Eq. (5). They only renormalize the behavior of single modes [28,31]. The generalization of our theory to the case when these terms are present is straightforward. We are thus left with two coupled equations of motion,

$$\ddot{u}_2 + \eta_2\dot{u}_2 + \omega_2^2 u_2 + 4\omega_2\gamma u_2 u_3 = 0;\ \ddot{u}_3 + \eta_3\dot{u}_3 + \omega_3^2 u_3 + 2\omega_3\gamma u_2^2$$
$$= 2\omega_3 F_{ef}\cos(\omega_d t), \tag{6}$$

where $F_{ef} = S_3 F_0$ and $\gamma = A_3 I_{22} K/2$. Note that the intermode detuning $\Delta = 2\omega_2 - \omega_3$ is tunable by u_{dc} and thus can be modulated by