Reprint of : Block-determinant formalism for an action of a multi-terminal scatterer

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A B S T R A C T

The scattering theory of electron transport allows for a compact and powerful description in terms of $\check{g}^2 = 1$ Green functions, the so-called circuit theory of quantum transport. A scatterer in the theory is characterized by an action, most generally a Keldysh one, that can be further used as a building bock of theories describing statistics of electron transport, superconducting correlations, time-dependent and interaction effects. The action is usually used in the form suitable for a two-terminal scatterer.

Here we provide a comprehensive derivation of a more general form of the action that is especially suitable and convenient for general multi-terminal scatterers. The action is expressed as a determinant of a block of the scattering matrix obtained by projection on the positive eigenvalues of the Green functions characterizing the reservoirs. We start with traditional Green function formalism introducing $\check{g}^2 = 1$ matrices and give a first example of multi-terminal counting statistics. Further we consider one-dimensional channels and discuss chiral anomaly arising in this context. Generalizing on many channels and superconducting situation, we arrive at the block-determinant relation. We give the necessary elaborative examples reproducing basic results of counting statistics and super-currents in multi-terminal junctions.

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1. Introduction

The well-established and refined culture of theoretical description of electron transport in bulk solids and heterostructures was based on field-theoretical methods [1] and Keldysh Green functions [2]. The pioneering works of Landauer and Buttiker [3,4] that unambiguously related electron transport and coherent scattering in micro-contacts have been regarded with suspicion: the genuine simplicity of their approach looked as a barbaric intrusion to a sophisticated domain. It took time to appreciate the idea that the electron resistance is in fact scattering. Once the appreciation of this revolutionary idea was in place, a fast research progress has revealed many facets of the universality of scattering approach and its relevance in the areas where its applicability was not at all obvious. The electric noise was understood in terms of scattering [5,6]. The sophistication came back when a state-of-the-art quantum calculation [7] has demonstrated that the whole statistics of electron transport is defined by the scattering matrix. The approach has been applied to superconducting contacts [8].

This also initiated research that combined Green function approaches with the notions of scattering and discrete elements

giving rise to a bunch of the so-called quantum circuit theories [9] that are indispensable for accessing full counting statistics, transmission distribution in complex scatterers, superconducting and spin transport in nanostructures. A starting point of this research was actually the paper of Buttiker and Beenakker about the noise in diffusive connector [10] that seemed wrong to the author and thus motivated him to prove the statement on more solid grounds. Quantum circuit theories possess a remarkable degree of universality. A two-terminal scatterer in this approach is always described by an action

$$S = \frac{1}{2} \sum_n \text{Tr}\left[\ln\left(1 + T_n \frac{\check{G}_1\check{G}_2 + \check{G}_2\check{G}_1 - 2}{4} \right)\right] \tag{1}$$

where n labels transport channels of the scatterer, T_n are corresponding transmission coefficients, while the matrices $\check{G}_{1,2}$ characterize the states of the leads, stem from the Green functions and satisfy $\check{G}_{1,2}^2 = 1$. The matrix structure as well as the role of the action conforms a concrete situation from the great variety where the relation can be applied. In case of circuit theory of transmission distribution [11] $\check{G}$ is a single-parametric 2×2 matrix and S is function to be minimized while in a theory encompassing interplay of Coulomb interaction and disorder [12] $\check{G}$ may represent supersymmetric σ-model quantum fields, matrix structure

includes time indices and the action is a part of a path integral weight.

In this paper, we address the generalization of Eq. (1) to the case of multiple terminals. Such generalization has hardly been discussed in the literature. One of the reasons for this is the fact that in a (quantum) circuit theory a multi-terminal scatterer can be readily modelled with two-terminal ones and at least a single node connected to the leads by means of these two-terminal scatterers. For instance, this is a way to multi-terminal counting statistics [13]. However, such approach is not general. On mean-field level, it disregards random phase factors accumulated in the course of the scattering in the node. The statistics of these random phase factors can in principle be obtained if proceeding beyond the mean-field level. However, this does not give an action for a concrete realization of these phase factors. Such action is especially important in the context of recent discovery of non-trivial topological phenomena in multi-terminal superconducting junctions [14].

A proposal for such generalization has been made in [15] in the context of understanding Fermi Edge singularity. In this paper, we provide a full and comprehensive derivation of this relation starting from the common textbook Green functions and extend it to the case of superconducting terminals. The result is expressed as a determinant of a block of the scattering matrix obtained by projection on the positive eigenvalues of the Green functions characterizing the reservoirs and is given by Eqs. (46) and (67).

The structure of the article is as follows. In Section 2 we discuss the Keldysh Green functions, its extension to counting statistics and give a single-state multiterminal example. In Section 3 we consider a transport channel connecting two reservoirs without scattering, compute Green functions and the action, recognize and heal a dangerous chiral anomaly. The generalization to many channels and scattering comes in Section 4. We introduce the superconducting reservoirs in Section 5 and generalize the action to this case in Section 6. Further we elaborate on two basic examples for the block-determinant formula obtained. In Section 7 we derive the full counting statistics for multi-terminal transport of the normal electrons. In Section 8 we perform the projection in superconducting case, consider the ground state energy of the junction and derive a useful formula for non-stationary superconducting current. We conclude in Section 9.

2. Green functions: general

We start our considerations with conventional definition [2] of Keldysh Green functions in terms of averages of fermion creation–annihilation operators $\Psi(t, X)$, with X being an element of a Hilbert space (for instance, space coordinate):

$$i\check{G}(X_1, X_2) = i\begin{bmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{bmatrix} = \langle \Psi_1 \Psi_2^\dagger \rangle \begin{bmatrix} \Theta_- & 1 \\ 0 & \Theta_+ \end{bmatrix}$$
$$- \langle \Psi_2^\dagger \Psi_1 \rangle \begin{bmatrix} \Theta_+ & 0 \\ 1 & \Theta_- \end{bmatrix}; \quad \Theta_\pm \equiv \Theta(\pm (t_1 - t_2)) \tag{2}$$

Here $\Psi_{1,2} \equiv \Psi(X_{1,2})$ and "check" denotes the matrix structure in the Keldysh index $i = \pm$. Let us specify to a general stationary none-quilibrium state where the density matrix is diagonal in the space of energy levels k and filling factor of this level is f_k. Since $\Psi(t) = \exp(-i\epsilon_k t)\Psi(0)$, the Green function is diagonal in the levels and reads

$$i\check{G}_k(t_1, t_2) = \exp(i\epsilon_k(t_2 - t_1))\left(\begin{bmatrix} \Theta_- & 1 \\ 0 & \Theta_+ \end{bmatrix} - f_k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right). \tag{3}$$

To get it in the energy representation, $\check{G}(\epsilon) \equiv \int dt e^{i\epsilon t} \check{G}(t, 0)$, we note

that

$$\int dt e^{i(\epsilon - \epsilon_k)t - \delta t}\Theta(t) = \frac{i}{\epsilon + i\delta - \epsilon_k} \equiv iG_R; \quad \text{Im } G_R = -i\pi\delta(\epsilon - \epsilon_k) \tag{4}$$

$$\int dt e^{i(\epsilon - \epsilon_k)t - \delta t}\Theta(-t) = \frac{-i}{\epsilon - i\delta - \epsilon_k} \equiv -iG_A;$$
$$\text{Im } G_A = i\pi\delta(\epsilon - \epsilon_k) \tag{5}$$

$$\int dt e^{i(\epsilon - \epsilon_k)t - \delta t} = 2\pi\delta(\epsilon - \epsilon_k) \equiv i(G_R - G_A) \tag{6}$$

where we have introduced the advanced and retarded Green functions not depending on the filling factors. With those, we can represent the Green function as

$$\check{G} = G_R \begin{bmatrix} -f & 1-f \\ -f & 1-f \end{bmatrix} + G_A \begin{bmatrix} f-1 & f-1 \\ f & f \end{bmatrix} \tag{7}$$

For proper description of the reservoirs we need what in old literature is called "Green function in coinciding points", $\check{G}(X, X)$. In fact, this concept has little to do with geometric proximity of the points: rather, it represents a Green function "averaged" over a big number of similar states of a quasicontinuous spectrum of the same energy. Let us generally define it as $\check{G}_{av} = \sum_k w_k \check{G}_k$, with w_p being some positive weights. The result of such averaging reads

$$\check{G}_{av} = -i\pi\nu \begin{bmatrix} 1 - 2f & 2(1-f) \\ -2f & 1-2f \end{bmatrix} \tag{8}$$

where $\nu \equiv \sum_k w_k \delta(\epsilon_k - \epsilon)$. For Green function in coinciding points, ν is the density of states. Its dependence on ϵ can be disregarded in all important cases. If the filling factors of the levels before the averaging depend on energy only, the filling factor $f(\epsilon)$ in the above relation naturally reproduces f of the levels. If not, $f(\epsilon)$ is a weighted average of those and effective filling factor of this group of the states if they are used as a reservoir.

The common Keldysh technique can be defined through the unitary evolution of the density matrix:

$$\rho(t) = T\exp(-iHt)\rho(-\infty)\tilde{T}\exp(iHt).$$

The extended Keldysh technique [9] is defined through a non-unitary evolution of the pseudo-density matrix with the Hamiltonians $H_\pm$ depending on the Keldysh index

$$\rho(t) = T\exp(-iH_- t)\rho(-\infty)\tilde{T}\exp(iH_+ t) \tag{9}$$

We define the action $\mathcal{S}$ in terms of the trace of this pseudo-density matrix after its evolution over a big interval of time $\mathcal{T}$, $e^{\mathcal{S}} = \text{Tr } \rho(\mathcal{T})$. Common application of extended Keldysh technique is full counting statistics of electron transfers [16]. In this case, $H_\pm = H \pm \chi/2I$, with I being the operator of current to a certain reservoir and the Fourier transform of $e^{\mathcal{S}}$ with respect to χ gives the probabilities of transferring N electrons during the time interval $\mathcal{T}$:

$$P_N = \int_0^{2\pi} \frac{d\chi}{2\pi} e^{i\chi N} e^{\mathcal{S}(\chi)}.$$

We will be interested in variations of the action. Let us assume that the Hamiltonians have been changed by a little addition $H_\pm \to H_{pm} + \sum_{ab} h_{ab}^\pm(t)\Psi_a^\dagger \Psi_b$. The corresponding variation of the action in the limit of small h is expressed in terms of the Green functions