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Nonlocal continuum analysis of a nonlinear uniaxial elastic lattice system under non-uniform axial load

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HIGHLIGHTS

- The static behavior of a nonlinear axial chain under distributed loading is examined.
- Exact analytical solutions based on Hurwitz zeta functions are presented.
- The nonlinear lattice possesses scale effects and possible localization properties in the absence of energy convexity.
- A nonlinear continuum elasticity model is developed to capture the main phenomena observed regarding the discrete axial problem.
- This associated continuum is an enriched gradient-based or nonlocal axial medium.

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ABSTRACT

The static behavior of the Fermi-Pasta-Ulam (FPU) axial chain under distributed loading is examined. The FPU system examined in the paper is a nonlinear elastic lattice with linear and quadratic spring interaction. A dimensionless parameter controls the possible loss of convexity of the associated quadratic and cubic energy. Exact analytical solutions based on Hurwitz zeta functions are developed in presence of linear static loading. It is shown that this nonlinear lattice possesses scale effects and possible localization properties in the absence of energy convexity. A continuous approach is then developed to capture the main phenomena observed regarding the discrete axial problem. The associated continuum is built from a continualization procedure that is mainly based on the asymptotic expansion of the difference operators involved in the lattice problem. This associated continuum is an enriched gradient-based or nonlocal axial medium. A Taylor-based and a rational differential method are both considered in the continualization procedures to approximate the FPU lattice response. The Padé approximant used in the continualization procedure fits the response of the discrete system efficiently, even in the vicinity of the limit load when the non-convex FPU energy is examined. It is concluded that the FPU lattice system behaves as a nonlocal axial system in dynamic but also static loading.

1. Introduction

In the 1950s, the study of oscillation in nonlinear lattices started with the Fermi-Pasta-Ulam (FPU) numerical experiment [1]. Fermi et al. accidentally found that the presence of a quadratic, a cubic, or even a piecewise linear additional term in the restoring force of an axial lattice with nearest neighbor interaction may be responsible for the so-called vibrations mode exchange phenomenon with a recurrence phenomenon, i.e. the possibility for the system to go back to its initial state after a recurrence time. The unexpected results of their work led to the development of the soliton theory by Kruskal and Zabusky [2] in the 1960s (see also Maugin, [3,4]). There are very few analytical solutions for nonlinear dynamics lattice problems; an exception may be mentioned for exponential interaction also called Toda lattice [5]. To date, to the authors' knowledge, analytical solutions for the dynamics of FPU lattice with polynomial nonlinearities are still not available in the literature. In this paper, we show that analytical solutions may be achieved in the static range, in presence of quadratic-type nonlinearity, which belongs to one of the three interaction configurations studied by Fermi et al. [1]. The quadratic-type nonlinearity in the restoring force of the lattice is associated with a cubic-type internal energy, by integration. Fermi et al. [1] initially

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considered a convex internal energy associated with a positive additional cubic term in the discrete energy of the nonlinear lattice system. In fact, convex and nonconvex type energy may be obtained from the energy equation postulated by Fermi et al. [1]. Nonconvex energy may have a strong physical support, as compared to the Lennard-Jones potential law for instance [6]. In the Lennard-Jones potential, the repulsive force contribution between atoms (Pauli repulsion) is combined with a long-range attractive force (van der Waals force) responsible for the loss of convexity. Therefore, the present paper explores both cases (convex and nonconvex energy) with different physical and engineering applications. The inertia effects will be neglected, and the static behavior of a nonlinear inelastic axial lattice system will be examined in presence of distributed loading such as a chain under its own weight. To the authors' knowledge, statics analytical solutions of the FPU lattice have not been reported in the literature. The paper of Gazis and Wallis [7] could be mentioned at this stage; they analytically investigated the static behavior of a linear lattice with nonlinear polynomial interaction concentrated at the border.

The paper also focuses on the possible definition of an equivalent axial continuum able to reproduce the main phenomena observed for the discrete lattice system. The definition of an equivalent continuum from a discrete one may be labeled as a continualization procedure. This question of relating discrete and continuous systems is an old one, and was already initiated by Lagrange [8] during the XVIIIth century. Lagrange [8] already showed the link between one-dimensional string or axial lattices with the associated one-dimensional asymptotic continua. During the XIXth century, Piola introduced some nonlocal integral type models from discrete microscale interactions, which may be expanded using higher-order gradient models [9,10]. Continualization procedures are based on various approximations of the discrete operators by some continuous ones via Taylor expansion or Padé approximants [2,11–15]. The so-called enriched continuum equivalent to the discrete one is sometimes called a quasi-continuum [12]. It is generally dependent on the truncated terms in the asymptotic expansion of the difference operators (see also the analysis of Zabusky and Kruskal within the dynamics of solitons [16]). The reader can refer to Rosenau [14], Palais [17] and Maugin [4] for a historical perspective on the link between the Fermi-Pasta-Ulam lattice model and the continualized wave propagation equation. Zabusky and Kruskal [16] used a Taylor expansion of the second-order finite difference operator arising in the discrete lattice up to the fourth-order spatial derivative. Benjamin et al. [18] and then Collins [12] proposed to use the inverse of the second-order finite difference operator, thus avoiding the use of fourth-order spatial operators. Padé approximants of the finite difference operators were introduced by Rosenau for FPU lattice systems [14] and are shown to be efficient for capturing the wave propagation in the dynamics of axial lattice without changing the highest spatial order of the wave equation. This rational expansion of the difference operator has been widely used for one-dimensional and two-dimensional media [11,13–15,19]. The quasi-continuum approximation of Toda lattice has been investigated by Hyman and Rosenau [20]. The same methodology can be applied in the context of static loading, as is shown in the paper. Most of the studies on these nonlinear lattices focused on axial wave propagation. In the present paper, we focus on the static problem of a nonlinear FPU axial chain, and we develop some possible analytical exact solutions for the lattice problem and approximate solutions for the equivalent continuum problem.

A similar approach has been followed by Triantafyllidis and Bardenhagen [21] who obtained numerical solutions for a nonlinear axial lattice under uniform axial load and with possible interaction with direct or other adjacent elements. Triantafyllidis and Bardenhagen [21] also investigated the possible continualization of the nonlinear lattice problem, from the displacement difference equation or directly from energy considerations. More recently, the computational community has been attracted by the numerical challenge of multiscale analysis, starting from lattice material configurations. Fish and Chen [22] have developed a multiscale approach for both static and dynamic molecular systems. Blanc et al. [23] provide an overview of mathematical results in multiscale computations. Recently, Carcaterra et al. [24] developed higher-order gradient continua from linear and nonlinear lattice interactions. Exact solutions of linear lattice problems can be attained from the resolution of linear finite difference equations (see for instance, Gazis and Wallis [7]; Mindlin [25] or more recently Challamel et al. [26]). Mindlin [19] described the behavior of a one-dimensional linear lattice, in the elastic range, taking into account interactions of three adjacent elements, and derived some possible gradient elasticity constitutive laws. The link between Eringen's nonlocal elasticity and lattice properties has been already outlined by Eringen [27] for axial wave propagation. It has been recently shown, using a continualization procedure, that Eringen's nonlocal elasticity [27] can also be used to describe the static and the dynamic behavior of linear lattice systems with nearest neighbor interaction. This has been shown to be relevant in linear elastic structural mechanics for axial, torsional and bending beam problems, for both static and dynamic applications (see Challamel et al., [26,28–30]). Eringen's nonlocality has also been shown to be efficient to capture the length scale effect of two-dimensional lattice systems (Zhang et al., [31,32]). The length scale of such a nonlocal model can be analytically adjusted from the size of the microscopic repetitive cell. The definition of an equivalent nonlocal medium for nonlinear lattice problems has probably been less investigated. Recently, the link between Discrete Damage Mechanics and nonlocal Continuum Damage Mechanics has been found for the bending problem of an elastic-damage axial chain [33]. The present paper shows, from the FPU lattice model, a possible relationship between nonlinear elastic lattices (Discrete Mechanics) and nonlocal nonlinear elasticity (Nonlocal Continuum Mechanics). A similar axial chain with damage irreversible constitutive law was examined in [34] where the localization phenomenon was investigated and related to nonlocal Continuum Damage Mechanics. The behavior of a nonlinear elastic bar under homogeneous stress state, whose material has a convex or a nonconvex energy property has been studied by Ericksen [35]. The latter investigated the stability of the possible multiple solutions of this system and evocated the possible finite discontinuous displacement field. This kind of discontinuity may be related to the concept of cohesive elasticity models based on an energetically-based displacement discontinuity evolution law. Cohesive elasticity models have been well developed by Del Piero and Truskinovsky [36,37], Marigo and Truskinovsky [38] or Charlotte et al. [39] based on theoretical and variational arguments. Gelli and Royer-Carfagni [40] derived zero-length cohesive laws from a one dimensional lattice models. However, these cohesive models mainly used some zero-length discontinuous displacement field. In the present paper, the continualization procedure leads to nonlocal elasticity models and a finite length cohesive model at the border (which is slightly different from the zero-length cohesive model). The present paper focuses on the nonlinear elastic (reversible) lattice model (different from the damage one) with both convex and nonconvex associated energy, with a connection to nonlocal, cohesive and nonlinear elasticity mechanics.

2. Static of the FPU lattice model

The FPU lattice model, composed of nonlinear elastic springs connected together, is examined in tension under a distributed