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Effect of a tunnel barrier on the scattering from a Majorana bound state in an Andreev billiard

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HIGHLIGHTS

- Scattering and time-delay matrices j.p.d.f. of certain random quantum dots is found.
- Heat current do not probe unpaired Majorana states if one mode is ballistic coupled.
- Ballistic coupling may not lift unpaired Majorana states dependence in dI/dV curve.
- The Majorana phase transition can be probed in nanowire-random quantum dot geometry.

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ABSTRACT

We calculate the joint distribution $P(S, Q)$ of the scattering matrix S and time-delay matrix $Q = -i\hbar S^{\dagger} dS/dE$ of a chaotic quantum dot coupled by point contacts to metal electrodes. While S and Q are statistically independent for ballistic coupling, they become correlated for tunnel coupling. We relate the ensemble averages of Q and S and thereby obtain the average density of states at the Fermi level. We apply this to a calculation of the effect of a tunnel barrier on the Majorana resonance in a topological superconductor. We find that the presence of a Majorana bound state is hidden in the density of states and in the thermal conductance if even a single scattering channel has unit tunnel probability. The electrical conductance remains sensitive to the appearance of a Majorana bound state, and we calculate the variation of the average conductance through a topological phase transition.

1. Introduction

The quantum states of particle and anti-particle excitations in a superconductor (Bogoliubov quasiparticles) are related by a unitary transformation, which means that they can be represented by a *real* wave function. In this so-called Majorana representation the $N \times N$ scattering matrix S at the Fermi level is real orthogonal rather than complex unitary [1]. Since the orthogonal group $O(N)$ is doubly connected, this immediately implies a twofold distinction of scattering problems in a superconductor: the subgroup $O_+(N) \equiv SO(N)$ of scattering matrices with determinant $+1$, connected to the unit matrix, is called *topologically trivial*, while the disconnected set $O_-(N)$ of scattering matrices with determinant -1 is called *topologically nontrivial*. In mathematical terms, the experimental search for Majorana bound states can be called a search for systems that have $\operatorname{Det} S = -1$. This search has been reviewed, from different perspectives, in Refs. [2–6].

If the scattering is chaotic the scattering matrix becomes very sensitive to microscopic details, and it is useful to develop a statistical description: rather than studying a particular S, one studies the probability distribution $P(S)$ in an ensemble of chaotic scatterers. This is the framework of random-matrix theory (RMT) [7–9]. The ensemble generated by drawing S uniformly from the unitary group $U(N)$, introduced by Dyson in the context of nuclear scattering [10], is called the circular unitary ensemble (CUE). Superconductors need a new ensemble. A natural name would have been the circular orthogonal ensemble (COE), but since that name is already taken for the coset $U(N)/O(N)$, the alternative name circular real ensemble (CRE) is used when S is drawn uniformly from $O(N)$. The RMT of the CRE, and the physical applications to Majorana fermions and topological superconductors, have been reviewed recently [11].

The uniformity of the distribution requires ideal coupling of the scattering channels to the continuum, which physically means that the discrete spectrum of a quantum dot is coupled to metal electrodes by ballistic point contacts. If the point contact contains a tunnel barrier, then $P(S)$ is no longer uniform but biased towards

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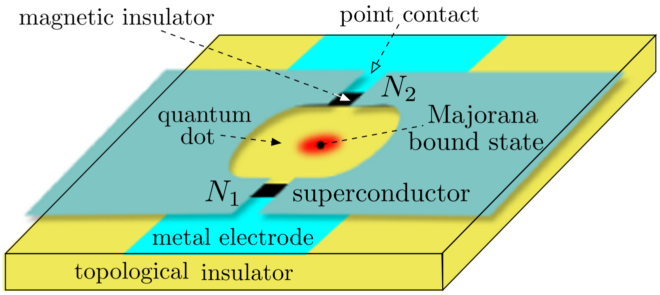

Fig. 1. Andreev billiard on the conducting surface of a three-dimensional topological insulator. The billiard consists of a confined region (quantum dot, mean level spacing δ_0) with superconducting boundaries, connected to metal electrodes by a pair of point contacts (supporting a total of $N = N_1 + N_2$ propagating modes). A magnetic insulator introduces a tunnel barrier in each point contact (transmission probability Γ per mode). A magnetic vortex may introduce a Majorana bound state in the quantum dot.

the reflection matrix r_B of the barrier. The modified distribution $P_{\rm Poisson}(S)$ is known [12–16], it goes by the name “Poisson kernel” and equals

$$P_{\rm Poisson}(S) \propto {\rm Det}(1 - r_B^\dagger S)^{1-N} \tag{1}$$

in the CRE [16].

In the present work we apply this result to the scattering (Andreev reflection) in a superconducting quantum dot (Andreev billiard), see Fig. 1. We focus in particular on the effect of a bound state at the Fermi level ($E=0$) in the quantum dot, a so-called Majorana zero-mode or Majorana bound state. In addition to the scattering matrix, which determines the thermal and electrical conductance, we consider also the time-delay matrix $Q = -i\hbar S^\dagger dS/dE$. The eigenvalues of Q are positive numbers with the dimension of time, that govern the low-frequency dynamics of the system (admittance and charge relaxation [17–19]). Moreover, the trace of Q gives the density of states and Q and S together determine the thermopower [20,21].

The joint distribution of S and Q is known for ballistic coupling [22–24], here we generalize that to tunnel coupling. The effect of a tunnel barrier on the time-delay matrix has been studied for complex scattering matrices [25,26], but not yet for real matrices. One essential distinction is that the tunnel barrier has no effect on the density of states in the CUE and COE, but it does in the CRE.

The outline of the paper is as follows. The next two sections formulate the scattering theory of the Andreev billiard and the appropriate random-matrix theory. Our key technical result, the joint distribution $P(S, Q)$, is given in Section 4. We apply this to the simplest single-channel case ($N=1$) in Section 5, where we obtain a remarkable scaling relation: for a high tunnel barrier (transmission probability $\Gamma \ll 1$) the distribution $P(\rho|\Gamma)$ of the density of states at the Fermi level is described by a one-parameter scaling function $F(x)$:

$$P(\rho|\Gamma) \propto \begin{cases} F(\Gamma\rho/4) & \text{with a Majorana bound state,} \\ F(4\rho/\Gamma) & \text{without a Majorana.} \end{cases} \tag{2}$$

The average density of states in the multi-channel case is calculated in Section 6. By relating the ensemble averages of Q and S we derive the relation

$$\langle\rho\rangle = \langle\rho\rangle_{\rm ballistic}\left(1 - \frac{2}{N\Gamma}\,{\rm Tr}\, r_B^\dagger[\langle S\rangle - r_B]\right), \tag{3}$$

for a mode-independent tunnel probability Γ. In the CUE and COE the average scattering matrix $\langle S\rangle$ is just equal to r_B, so $\langle\rho\rangle$ remains equal to its ballistic value $\langle\rho\rangle_{\rm ballistic}$, but the CRE is not so constrained.

Applications to the thermal conductance g and the electrical (Andreev) conductance g_A follow in Sections 7 and 8, respectively. For ballistic coupling it is known that $P(g)$ is the same with or without the Majorana bound state [27]. (This also holds for $P(\rho)$ [23].) In the presence of a tunnel barrier this is no longer the case, but we find that the Majorana bound state remains hidden if even a single scattering channel has $\Gamma=1$. The distribution of g_A, in contrast, is sensitive to the presence or absence of the Majorana bound state even for ballistic coupling [28]. The way in which $P(g_A)$ changes as we tune the system through a topological phase transition, at which a Majorana bound state emerges, is calculated in Section 9. We conclude in Section 10.

In the main text we focus on the results and applications. Details of the calculations are moved to the Appendices. These also contain more general results for other RMT ensembles, with or without time-reversal and/or spin-rotation symmetry. (Both symmetries are broken in the CRE.)

2. Scattering formulation

Fig. 1 shows the scattering geometry, consisting of a superconducting quantum dot (Andreev billiard) on the surface of a topological insulator, connected to normal metal electrodes by point contacts. The Hamiltonian H of the quantum dot is related to the energy-dependent scattering matrix $S(E)$ by the Mahaux–Weidenmüller formula [29],

$$S(E) = \frac{1 - i\pi W^\dagger(E-H)^{-1}W}{1 + i\pi W^\dagger(E-H)^{-1}W} = 1 - 2\pi i W^\dagger(E - H + i\pi WW^\dagger)^{-1}W. \tag{4}$$

The $M \times N$ matrix W couples the M energy levels in the quantum dot (mean level spacing δ_0) to a total of $N \ll M$ propagating modes in the point contact.

We assume that degeneracies are broken by spin–orbit coupling in the topological insulator in combination with a magnetic field (perpendicular to the surface). All degrees of freedom are therefore counted separately in N and M, as well as in δ_0. The electron–hole degree of freedom is also included in the count, but we leave open the possibility of an unpaired Majorana fermion — a coherent superposition of electron and hole quasiparticles that does not come with a distinct antiparticle. An odd level number M indicates the presence of a Majorana bound state in the quantum dot, produced when a magnetic vortex enters [30]. An odd mode number N signals a propagating Majorana mode in the point contact, allowed by a π-phase difference between the superconducting boundaries [31].

The N modes have transmission probability $\Gamma_n \in [0, 1]$ per mode. We neglect the energy dependence of the Γ_n's, which is applicable if the coupling is via a high and narrow potential barrier (realized, for example, by a magnetic insulator in the point contact). If we choose a basis such that the coupling matrix W has only nonzero elements on the diagonal, it has the explicit form [32]

$$W_{mn} = w_n\delta_{mn}, \quad 1 \le m \le M, \quad 1 \le n \le N,$$

$$|w_n|^2 = \frac{M\delta_0\kappa_n}{\pi^2}, \quad \kappa_n = \frac{1 - r_n}{1 + r_n}, \quad r_n^2 = 1 - \Gamma_n. \tag{5}$$

Notice that the tunnel probability Γ_n determines the reflection amplitude $r_n \in [-1, 1]$ up to a sign. The conventional choice is to take $r_n \ge 0$, when $\kappa_n = \kappa_n^+$ can be written as

$$\kappa_n^+ = \frac{1}{\Gamma_n}(2 - \Gamma_n - 2\sqrt{1 - \Gamma_n}). \tag{6}$$

Alternatively, if $r_n \le 0$ one has $\kappa_n = \kappa_n^-$ given by