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# Geometric phase and Pancharatnam phase induced by light wave polarization

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ABSTRACT

We use the quantum kinematic approach to revisit geometric phases associated with polarizing processes of a monochromatic light wave. We give the expressions of geometric phases for any, unitary or non-unitary, cyclic or non-cyclic transformations of the light wave state. Contrarily to the usually considered case of absorbing polarizers, we found that a light wave passing through a polarizer may acquire in general a nonzero geometric phase. This geometric phase exists despite the fact that initial and final polarization states are in phase according to the Pancharatnam criterion and cannot be measured using interferometric superposition. Consequently, there is a difference between the Pancharatnam phase and the complete geometric phase acquired by a light wave passing through a polarizer. We illustrate our work with the particular example of total reflection based polarizers.



## 1. Introduction

The concept of geometric phase naturally arises for polarized light in optics. In 1956, Pancharatnam [1] studied how the phase of polarized light changes after a cyclic evolution of its polarization. He found that light wave acquires, in addition to the usual phase associated with the optical path, a geometric phase depending only on the relative *loci* of the polarization states on the Poincaré sphere. Later, Berry [2] developed the concept of geometric phase for dynamical quantum systems with cyclic adiabatic unitary evolutions and showed its similarity with the Pancharatnam phase in optics [3]. The existence of geometric phase has been also demonstrated for nonunitary and noncyclic evolutions [4,5], and recently for open quantum systems [6–10]. Many experiments [11–18] have provided evidence for geometric phase in the context of polarized light. Along a given path, closed or not, on the Poincaré sphere, bringing the polarization from a state $|1\rangle$ to a state $|2\rangle$, the light wave state acquires a geometric phase which is equal to minus half of the solid angle enclosed by the effectively followed path and the geodesic connecting states $|1\rangle$ and $|2\rangle$ [5]. If the path coincides with the geodesic then no geometric phase is gained. Since any cyclic path on the Poincaré sphere is at least a concatenation of two geodesics and is therefore by itself not a geodesic, a light wave along such a path acquires *de facto* a nonzero geometric phase. This property has been widely used in the above cited experiments where cyclic evolution of the polarization state was usually achieved using retarders (unitary transformations) [11–14,16], polarizers (non-unitary transformations) [16–18], and both [11,15].

When a retarder (*e.g.* a wave plate) is used on a light wave, its polarization follows then a piece of circle on the Poincaré sphere and a geometric phase is consequently acquired (except if the piece of circle is a piece of great circle of length less than $\pi$). As far as we know, in the literature, the action of a polarizer is considered to not introduce geometric phase since it is considered to project the light wave polarization from a state onto another following a geodesic [11,15–18]. This is effectively true for the case of absorbing polarizers. We find that this is no longer true if one considers total reflection based polarizers since the path followed by the polarization is no more a geodesic but a loxodrome. More generally, we show that even if a light wave state $|1\rangle$ is projected onto another state $|2\rangle\langle 2|1\rangle$, for example by means of a polarizer, a nonzero geometric phase can be acquired by the light wave. At the end of such a transformation the acquired geometric phase is exactly compensated by the acquired dynamic phase in such a way that the total phase has no memory of these two phases. As a consequence interferometry measurements are not able to capture possible geometric phase acquired during a state projection.

However the Pancharatnam phase [1], which is a kind of geometric phase, can be measured using interferometry experiments. As reminded by de Vito and Levrero [19], we can consider that a light wave acquires a Pancharatnam phase if in the Hilbert

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space the light wave state is projected successively onto states the polarizations of which marked out a cyclic path on the Poincaré sphere. The use of successive polarizers ensures that the light wave state is successively projected and then that a Pancharatnam phase is indeed measured [16–18]. We show that the Pancharatnam phase is not the complete geometric phase indeed acquired by a light wave since the Pancharatnam phase does not take into account the geometric phase possibly acquired during the projection processes.

In this paper, we also provide the most general expression of the geometric phase acquired by a light wave experiencing a polarizing transformation. We address particularly the case of the possible geometric phase acquired by a light wave passing through a total reflection based polarizer. The paper is organized as follows: In Section 2, we define the mathematical formalism describing the light wave state and its polarization. Using the quantum kinematic approach [20], in Section 3, we describe the geometric phase and the modulus of the degree of coherence as the gauge invariant quantities associated with a non-unitary evolution of the light wave state. In Section 4, we classify any light wave state transformation induced by a polarizing element in terms of $SL(2,\mathbb{C})$ transformations. In Section 5, first, we revisit the case of unitary transformations corresponding to retarders or to media with optical activity and the case of nonunitary transformation corresponding to absorbing polarizing elements. Then, we show that, in the case of nonunitary transformation combining both differential attenuation and differential dephasing a light wave does acquire a geometric phase and we derive its expression. In Section 6, we derive for the sake of completeness the total phase and the dynamic phase acquired by a light wave passing through a polarizing device using the model [21]. In the frame of this model we express the Pancharatnam criterion [1]. In Section 7, we apply the results derived in Section 5 for general polarizing elements to the specific case of polarizers. We found that a polarizer in general does induce a nontrivial geometric phase and we derive its expression for the case of a total reflection based polarizer. This geometric phase exists despite the fact that the initial and final polarization states are *in phase* according to the Pancharatnam criterion and is a direct reminiscence of the evanescent component of the electromagnetic field inside the polarizer. In the limit where differential absorption is predominant over birefringence (*e.g.* in polaroid films), we retrieve as expected a zero geometric phase. In Section 8, we discuss the difference between the Pancharatnam phase and the geometric phase acquired by a light wave.

## 2. Polarization, space of rays, and Poincaré sphere

A polarized light wave may be described by a vector $|\psi\rangle$ lying in a two dimensional complex Hilbert space $\mathcal{H}$. Such a vector $|\psi\rangle$ may be written as

$$|\psi\rangle=\sqrt{I}\,e^{i\Phi}\left(\cos\frac{\theta}{2}|0\rangle+e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right) \tag{1}$$

where $I=\langle\psi|\psi\rangle\in\mathbb{R}^+$ is the light wave intensity, $\Phi\in[0,2\pi[$ a global phase, $\phi\in[0,2\pi[$ a relative phase, $\theta\in[0,\pi]$ the polar angle, and $\{|0\rangle,|1\rangle\}$ an orthonormal basis of $\mathcal{H}$. The vectors $|0\rangle$ and $|1\rangle$ represent *e.g.* the normalized state with circular right-handed polarization and that with circular left-handed polarization respectively.

The polarization of the light wave $|\psi\rangle$ depends only on the ellipticity angle $\chi$ and on the azimuthal angle $\Psi$ [22] which are directly related to the polar angle $\theta=\pi/2-2\chi$ and to the relative phase $\phi=2\Psi$. So, two light waves $|\psi\rangle$ and $|\psi'\rangle=a|\psi\rangle$, where $a$ is a complex factor, share the same polarization. We say that $|\psi\rangle$ and $|\psi'\rangle$ are equivalent, *i.e.* $|\psi'\rangle\sim|\psi\rangle$, in the sense that it is possible to convert one of these wave to the other by using a complex scale transformation. Let us then define the space $\mathcal{R}$ of unit rays by $\mathcal{R}=\mathcal{H}/\sim\ =\{\rho=I^{-1}|\psi\rangle\langle\psi|\,|\,|\psi\rangle\in\mathcal{H}\}$. An element $\rho$ belonging to $\mathcal{R}$ may be written as

$$\rho=\frac{1}{2}(\sigma^0+\vec{S}\cdot\vec{\sigma})\equiv\rho_{\vec{S}} \tag{2}$$

where $\vec{\sigma}$ is a three dimensional vector whose components are the Pauli matrices $\{\sigma^i\}_{i=1,2,3}$ and where $\sigma^0$ is the $2\times2$ identity matrix. Any projector $\rho$ is associated with a unique normalized Stokes vector $\vec{S}=\sin\theta\cos\phi\,\vec{e}^{\,1}+\sin\theta\sin\phi\,\vec{e}^{\,2}+\cos\theta\,\vec{e}^{\,3}$. The set of the endpoints of all the normalized Stokes vectors defines the Poincaré sphere $\mathcal{S}^2$. Each $\vec{S}$ vector is in bijective relation with a point in the space of rays $\mathcal{R}$, *i.e.* with a projector belonging to $\mathcal{R}$. So, the unit Poincaré sphere $\mathcal{S}^2$ is isomorphic to the space of unit rays $\mathcal{S}^2\sim\mathcal{R}$. The set of vectors $\{|\psi'\rangle=a|\psi\rangle,\ a\in\mathbb{C}\}$ corresponds to a unique projector $\rho_{\vec{S}}$ (2) and consequently corresponds to a unique normalized Stokes vector $\vec{S}$. A wave $|\psi\rangle$ as defined in Eq. (1) may be then represented, *modulo a global complex factor*, by a point in the space of unit rays $\mathcal{R}$ or equivalently by a point on the Poincaré sphere $\mathcal{S}^2$. The circular right(left)-handed polarization state $|0\rangle$ ($|1\rangle$), corresponds to $\theta=0$ ($\theta=\pi$), *i.e.* to the north (south) pole of the Poincaré sphere. Linear polarization states correspond to vectors of $\mathcal{H}$ with $\theta=\pi/2$, or equivalently correspond to points of the Poincaré sphere equator.

## 3. Local gauge invariance

Passing through an optical device, the state $|\psi\rangle$ of a light wave evolves in the Hilbert space $\mathcal{H}$ along a curve $\mathcal{C}=\{|\psi(s)\rangle\in\mathcal{H}\,|\,s\in[s_1,s_2]\subset\mathbb{R}\}\subset\mathcal{H}$. Let us now define another curve $\mathcal{C}'$ the elements of which are related to the elements of $\mathcal{C}$ by a *local gauge transformation*, $|\psi'(s)\rangle=a(s)|\psi(s)\rangle$. Here, $a(s)$ is a smooth nonzero complex function of $s\in[s_1,s_2]$. Comparing $\langle\psi'(s)|d/ds|\psi'(s)\rangle$ with $\langle\psi(s)|d/ds|\psi(s)\rangle$, it is possible to construct the following complex gauge invariant expression [20]:

$$\frac{\langle\psi(s_1)\,|\,\psi(s_2)\rangle}{\langle\psi(s_1)\,|\,\psi(s_1)\rangle}\exp\left(-\int_{s_1}^{s_2}ds\frac{\langle\psi(s)\,|\,\dot{\psi}(s)\rangle}{\langle\psi(s)\,|\,\psi(s)\rangle}\right). \tag{3}$$

Here the dot denotes the differentiation with respect to the parameter $s$. Let us define the projection map $\pi:\mathcal{H}\to\mathcal{R}$ such as, for all $a\in\mathbb{C}$, $\pi(a|\psi\rangle)=\pi(|\psi\rangle)=\rho\in\mathcal{R}$. Since the curves $\mathcal{C}'$ and $\mathcal{C}$ are related by a gauge transformation, $\mathcal{C}\sim\mathcal{C}'$, they share the same projected curve image $\mathrm{C}=\pi(\mathcal{C})=\pi(\mathcal{C}')$ in the space of unit rays $\mathcal{R}$. As expression (3) is gauge invariant, it is a functional of the curve C and, its modulus $\iota_g[\mathrm{C}]$ and complex argument $\phi_g[\mathrm{C}]$ are also gauge invariant functionals of the curve C. The modulus of Eq. (3) can be written in the following form:

$$\iota_g[\mathrm{C}]=\frac{|\langle\psi(s_1)\,|\,\psi(s_2)\rangle|}{\sqrt{I(s_1)I(s_2)}}=\sqrt{\mathrm{Tr}(\rho(s_1)\rho(s_2))} \tag{4}$$

which is also the modulus of the complex degree of coherence $\gamma_{12}(0)=\iota_g[\mathrm{C}]e^{i\arg\langle\psi(s_1)|\psi(s_2)\rangle}$. Hence the modulus of the interference term between the two normalized wave states $(1/\sqrt{I(s_1)})|\psi(s_1)\rangle$ and $(1/\sqrt{I(s_2)})|\psi(s_2)\rangle$ is a geometric invariant. The complex argument of Eq. (3) is the geometric phase [20] associated with the curve $\mathrm{C}\subset\mathcal{R}$

$$\begin{aligned}\phi_g[\mathrm{C}]&=\arg\langle\psi(s_1)\,|\,\psi(s_2)\rangle-\mathrm{Im}\int_{s_1}^{s_2}ds\frac{\langle\psi(s)\,|\,\dot{\psi}(s)\rangle}{\langle\psi(s)\,|\,\psi(s)\rangle}\\&\equiv\phi_t[\mathcal{C}]-\phi_d[\mathcal{C}].\end{aligned} \tag{5}$$