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Anderson localization in a two-dimensional random gap model

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HIGHLIGHTS

- Our work studies an Anderson transition in a 2D electron gas with a random gap.
- A diffusive regime exists for weak scattering and a localized regime for strong scattering.
- The numerical results are supported by a convergent strong coupling expansion in the localized regime.
- The exponent of the localization length varies from 1.3 to 1.5 and depends on model parameters.

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ABSTRACT

We study the properties of the spinor wavefunction in a strongly disordered environment on a two-dimensional lattice. By employing a transfer-matrix calculation we find that there is a transition from delocalized to localized states at a critical value of the disorder strength. We prove that there exists an Anderson localized phase with exponentially decaying correlations for sufficiently strong scattering. Our results indicate that suppressed backscattering is not sufficient to prevent Anderson localization of surface states in topological insulators.

1. Introduction

The classical approach to randomly scattered particles leads to diffusion, where random scattering originates either from particle–particle collisions (e.g., in a gas) or from collisions with (static) impurity scatterers. In quantum systems, however, diffusion appears only for weak disorder whereas it is destroyed due to Anderson localization at stronger randomness [1,2]. This effect is particularly strong in low-dimensional systems, such as two-dimensional graphene sheets or the surface of topological insulators. The scaling approach to generic random scattering [2] states that diffusion is entirely suppressed by Anderson localization for dimension $d \leq 2$. On the other hand, it has been argued that Anderson localization is prevented on the surface of topological insulators due to suppressed backscattering [3,4].

Inspired by the recent observation of metallic behavior (i.e. diffusive or even ballistic transport) in disordered two-dimensional systems (graphene) [5,6], a general discussion of diffusion and localization of quantum particle is required, which takes into account a spinor structure of the wavefunction. Two possibilities have been considered, namely ballistic transport for finite systems [7,8] and diffusive transport for infinite systems [9]. Diffusion is related to long ranged correlations, which is usually caused by spontaneous symmetry breaking [9,10]. This behavior might be restricted to the regime of weak scattering, since strong scattering is capable to localize particles. The aspect of weak localization is ignored here on purpose because it has its own problems [11,12]. This will be discussed in a separate paper. Instead, we will focus in the following mostly on the case of strong scattering. This is motivated by recent numerical studies, which have indicated that there is a transition to a localized phase at sufficiently strong disorder [13,14]. Here we will analyze details of the transition in terms of the scaling behavior of the localization length for strips of finite width. Moreover, the infinite system will be treated analytically within a strong scattering expansion. The latter provides a rigorous proof for exponential localization, supporting the numerical results at strong disorder. We study a random gap model with linear spectrum (2D Dirac fermions), but our methods can be easily applied to other systems as well.

2. Model

We consider the surface Hamiltonian of a topological insulator with bulk inversion symmetry of momentum $\mathbf{k}$ [3,14–16]

$$H = \begin{pmatrix} h(\mathbf{k}) & 0 \\ 0 & h^*(-\mathbf{k}) \end{pmatrix}, \quad h(\mathbf{k}) = \hbar \begin{pmatrix} C+M-(D+\delta)k^2 & v_F(k_x+ik_y) \\ v_F(k_x-ik_y) & C-M-(D-\delta)k^2 \end{pmatrix} \quad (1)$$

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This Hamiltonian consists of a pair of massive Dirac Hamiltonians $h(\mathbf{k})$, $h^*(-\mathbf{k})$. It should be noticed that this Hamiltonian reads in coordinate space

$$H = \begin{pmatrix} h & 0 \\ 0 & h^T \end{pmatrix}$$

with the matrix transposition ^{T}. We include disorder by a random variable M with mean $\overline{m}$. For our numerical transfer-matrix calculation we use a box distribution with width W. For simplicity we choose the Dirac point, where $C=0$ and $D=0$. The main feature is that there are two bands that touch each other at a spectral node $k=0$ if $M=0$, whereas $M\neq 0$ opens a gap $\Delta = 2|M|$. Thus, a random M creates a random gap. Our aim is to calculate the localization length Λ of the eigenstate ψ at energy $E=0$ which satisfies $h\psi = 0$ and the transition probability of a moving particle. The two block Hamiltonians $h(\mathbf{k})$, $h^*(-\mathbf{k})$ act on two separate spaces with the same localization properties. Therefore, it is sufficient to study just one of them.

2.1. Localization length

The localization length Λ of the eigenstates of Hamiltonian (1) can be calculated numerically within a transfer-matrix approach. For this purpose the continuous Hamiltonian must be discretized in space (cf. Appendix A). Then the transfer-matrix T_l of the eigenvalue problem $\psi_{l+1} = h^Y\psi_l + h^D\psi_{l-1}$ (cf. Eqs. (27) and (28)) reads

$$T_l = \begin{pmatrix} h^Y & h^D \\ 1 & 0 \end{pmatrix}, \tag{2}$$

which enables us to evaluate the Lyapunov exponents of the wavefunction [17,18]. With the initial values ψ_0 and ψ_1 the iteration of Eq. (28) provides the wavefunction ψ_L at site L by applying the product matrix

$$M_L = \prod_{l=1}^{L} T_l. \tag{3}$$

For a random Hamiltonian this is a product of random matrices that satisfies Oseledec's theorem [19]. The latter states that there exists a limiting matrix

$$\Gamma = \lim_{L\to\infty} (M_L^\dagger M_L)^{1/2L}. \tag{4}$$

The eigenvalues of Γ are usually written as a diagonal matrix with exponential functions $\exp(\gamma_i)$, where γ_i is the Lyapunov exponent (LE). Adapting the numerical algorithm described in Ref. [18], the whole Lyapunov spectrum can be calculated and the smallest LE is identified with the inverse localization length $1/\Lambda$ [17]. Λ increases with the system width M according to a power law $\Lambda \propto M^\alpha$, where $\alpha > 1$ ($\alpha < 1$) in the regime of extended (localized) states, and $\alpha = 1$ in the critical regime. For the exponentially localized regime we expect $\Lambda \propto$ const. According to the one-parameter scaling theory by MacKinnon [20], the normalized localization length $\tilde{\Lambda} = \Lambda/M$, being a function of disorder strength W and system width M, depends only on a single parameter

$$\tilde{\Lambda}(M, W) = f(\xi(W)/M), \tag{5}$$

where ξ is a characteristic length of the system generated by disorder. Thus, any change of disorder strength W can be compensated by a change of the system width M. If there is a scale-invariant point W_c we can expand $\tilde{\Lambda}$ in its vicinity by assuming a power law with critical exponent ν of the correlation length as $\xi = |W - W_c|^{-\nu}$. Then we have [18]

$$\ln\tilde{\Lambda} = \ln\tilde{\Lambda}_c + \sum_{s=1}^{S} A_s(|W-W_c|M^{1/\nu})^s = \ln\tilde{\Lambda}_c + \sum_{s=1}^{S} A_s\left(\frac{\xi}{M}\right)^{-s/\nu}. \tag{6}$$

2.2. Transition probability

The motion of a quantum particle from site $\mathbf{r}'$ to site $\mathbf{r}$ during the time t is described by the transition probability

$$P_{\mathbf{r}\mathbf{r}'}(t) = |\langle\mathbf{r}|\exp(-iHt)|\mathbf{r}'\rangle|^2. \tag{7}$$

If we assume that $P_{\mathbf{r}\mathbf{r}'}(t)$ describes diffusion, we can obtain the mean square displacement with respect to $\mathbf{r}' = 0$ from the diffusion equation

$$\langle r_k^2\rangle = \sum_{\mathbf{r}} r_k^2 P_{\mathbf{r},0}(t) = Dt, \tag{8}$$

which, after applying a Laplace transformation, becomes

$$\sum_{\mathbf{r}} r_k^2 \int_0^\infty P_{\mathbf{r},0}(t) e^{-\varepsilon t}\, dt = \frac{D}{\varepsilon^2}. \tag{9}$$

Using the Green's function $G_{\mathbf{r}\mathbf{r}'}(z) = (H-z)^{-1}_{\mathbf{r}\mathbf{r}'}$, we obtain for large distances $|\mathbf{r}-\mathbf{r}'|$ and $\varepsilon \sim 0$

$$\int_0^\infty P_{\mathbf{r}\mathbf{r}'}(t)e^{-\varepsilon t}\, dt \sim \int_{E_0}^{E_F} \langle|G_{\mathbf{r}\mathbf{r}'}(E+i\varepsilon)|^2\rangle_d\, dE = \int_{E_0}^{E_F} \langle G_{\mathbf{r}\mathbf{r}'}(E+i\varepsilon)G_{\mathbf{r}'\mathbf{r}}(E-i\varepsilon)\rangle_d\, dE, \tag{10}$$

where $\langle\ldots\rangle_d$ is the average with respect to disorder that is causing scattering. E_0 is the lower band edge and $\mathrm{Tr}_4(\ldots)$ is the trace with respect to the 4 spinor components. The second equation is due to the fact that the Hamiltonian is Hermitean. Then we get with $\mathbf{r}' = 0$ from Eq. (10) for the diffusion coefficient at the energy E

$$D(E) \sim \lim_{\varepsilon\to 0} \varepsilon^2 \sum_{\mathbf{r}} r_k^2 \langle G_{\mathbf{r}0}(E+i\varepsilon)G_{0\mathbf{r}}(E-i\varepsilon)\rangle_d \tag{11}$$

with $D = \int_{E_0}^{E_F} D(E)\, dE$ in Eq. (9).

According to Eq. (9), diffusion requires a long range correlation for small ε in Eq. (10). Anderson localization, on the other hand, is characterized by an exponentially decaying correlation. A natural approach to study the latter for strong randomness would be a hopping expansion in Eq. (11). Unfortunately, such an expansion is plagued by poles on both sides of the real axis. This problem can be avoided if we focus on the most relevant contributions of the randomly fluctuating product of Green's functions $G_{\mathbf{r},\mathbf{r}'}(i\varepsilon)G_{\mathbf{r}',\mathbf{r}}(-i\varepsilon)$. They are associated with the underlying chiral symmetry. These fluctuations have been studied previously in Ref. [21], where the large scale behavior was found to be associated with the Grassmann integral

$$K_{\mathbf{r}\mathbf{r}'} = \langle G_{\mathbf{r}0}(E+i\varepsilon)G_{0\mathbf{r}}(E-i\varepsilon)\rangle_d \approx K_0 \int \varphi_{\mathbf{r}}\varphi'_{\mathbf{r}} J\mathcal{D}[\varphi,\varphi'] \tag{12}$$

with $\mathcal{D}[\varphi,\varphi'] = \prod_{\mathbf{r}} d\varphi\, d\varphi'$ and with the Jacobian

$$J = \frac{1}{detg(H_0 + i\varepsilon + i\eta\hat{U}^2)}, \quad H_0 = \langle H\rangle, \quad \hat{U}_{\mathbf{r}} = \begin{pmatrix} \mathbf{1}+2\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}} & -2\varphi_{\mathbf{r}}\sigma_1 \\ -2\varphi'_{\mathbf{r}}\sigma_1 & \mathbf{1}-2\varphi_{\mathbf{r}}\varphi'_{\mathbf{r}} \end{pmatrix}. \tag{13}$$

The Jacobian appears since we have restricted the integration over randomness to those degrees of freedom which are associated with a global symmetry of the system. It is written in terms of a graded determinant *detg*, where the latter is expressed by conventional determinants in the relation

$$detg\begin{pmatrix} A & \Theta \\ \overline{\Theta} & B \end{pmatrix} = \frac{\det(A)}{\det(B)}\det(\mathbf{1} - \Theta B^{-1}\overline{\Theta}A^{-1}).$$

The parameter η is the scattering rate, which can be considered as an external parameter that is either calculated in self-consistent Born