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On the micromechanical definition of macroscopic strain and strain-rate tensors for granular materials

Q.-C. He *

Southwest Jiaotong University, School of Mechanical Engineering, Chengdu 610031, China
Université Paris-Est, Laboratoire de Modélisation et Simulation Multi Echelle, MSME UMR 8208 CNRS, 5 Boulevard Descartes, 77454 Marne-la-Vallée Cedex 2, France

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ABSTRACT

Studied is the three-dimensional definition of macroscopic kinematical measures for a representative assembly element (RAE) of a dense granular material, which is assimilated to a representative volume element (RVE) filled with a continuum medium. The geometrical transition from an RAE to an RVE is constructed by means of the Voronoi and Delaunay tessellations associated with the grain centers. After establishing the compatibility conditions that the grain center displacements must satisfy, explicit micromechanical expressions are derived for the macroscopic deformation and velocity gradient tensors. Using these expressions, various macroscopic strain and strain-rate tensors are defined as in continuum mechanics.

1. Introduction

In mechanics of granular materials, a subject of long standing is to correlate the overall mechanical behavior of a granular material to the geometry, spatial distribution and mechanical behavior of the constituent grains. The discrete, or discontinuous, nature of granular materials makes them rather different from the inhomogeneous materials considered within the usual framework of micromechanics [see, e.g., 17,20,32]. For this reason, some basic micromechanical concepts and relationships, which are well-established in the spirit of continuum mechanics, fail to be valid for granular materials. Among these basic concepts and relationships, the definition of macroscopic strain and strain-rate tensors has indirectly or directly attracted attention for long time but is still an open problem [see, e.g., 1–7,9–14,26–28,34,37]. Lack of a general method for defining macroscopic strain and strain-rate tensors currently constitutes a source of difficulties encountered not only in the development of micromechanics of granular materials [8] but also in the interpretation of experimental results [9] and in the computational mechanics of granular materials [6,28]. The present paper attempts to go a step further towards solving this fundamental problem.

In micromechanics, the definition of macroscopic deformation and stress measures is primarily based on the requirement that they be completely determined by the boundary data of a representative volume element. This is because theories or experimental determinations of the mechanical behavior of inhomogeneous materials rest ultimately on the overall response of such an element via measured loads and displacements on its surface [see, e.g., 21,23,31]. For the inhomogeneous materials (polycrystals, composites, etc.) usually considered in micromechanics, it turns out that the volume averages of the Cauchy stress and infinitesimal strain tensors in the case of infinitesimal transformations, and the volume averages of the first Piola–Kirchhoff stress and deformation gradient tensors in the case of finite transformations, satisfy the aforementioned requirement and behave as suitable measures for the macroscopic deformation and stress. The reason that the volume averages of these tensor fields are uniquely dependent on surface data alone is that they are assumed to verify, respectively, the equilibrium and compatibility equations. This simple observation is the key to understanding why an explicit micromechanical expression for the macroscopic stress of a system of grains was already available in the famous treatise of Love [29] (pp. 616–619) [see also 10,25,26,38] while an equivalent commonly accepted for the macroscopic strain is still lacking. In fact, for a (cohesionless) granular material, i.e., an assembly of small grains whose interactions obey the unilateral contact law [see, e.g., 19,30], the equilibrium equations for the inter-granular contact forces are classical and, on the other hand, the compatibility equations for the inter-granular relative displacements are not available in an

* Address: Université Paris-Est, Laboratoire de Modélisation et Simulation Multi Echelle, MSME UMR 8208 CNRS, 5 Boulevard Descartes, 77454 Marne-la-Vallée Cedex 2, France. Tel.: +33 160957786.

E-mail address: qi-chang.he@univ-paris-est.fr

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explicit general form. This prevents us from relating boundary to internal grain displacements and, thus, from obtaining a micromechanical expression for the macroscopic strain.

After identifying the main obstacle to the micromechanical definition of macroscopic strain and strain-rate tensors for granular materials, we tackle it by considering a simple model. This model consists in geometrically replacing the constituent grains of a representative assembly element of a statistically homogeneous dense granular material by their centers, and taking the domain generated and partitioned by the Delaunay tessellation as the domain of a representative volume element filled with a continuum medium. Inspired by a work of Kruyt and Rothenburg [26] and using the Gauss (divergence) theorem, we establish the three-dimensional compatibility equations for the grain center transformations. By means of these equations, we further derive explicit micromechanical expressions for the macroscopic deformation and velocity gradient tensors. Thereby, various macroscopic strain and strain-rate tensors can be defined as in continuum mechanics.

The paper is organized as follows. In the next section, the title problem is formulated in such a manner as to make appear the main difficulties. In Section 3, the formulated problem is approached on the basis of the aforementioned simple model. In Section 4, a few concluding remarks are given.

2. Setting of the problem

In what follows, a granular material refers to a medium consisting of a countable number of rigid or deformable solid grains (or particles) whose interactions comply with the *unilateral contact law*, i.e., with a kinematic condition of *impenetrability*, a static condition of *no tension* and a mixed condition of *complementarity* [see, e.g., 19,30]. In what follows, we assume that the granular material under consideration is *statistically homogeneous* so as to admit a *representative assembly element* (RAE). In addition, we make the assumption that the granular material is *dense*. Precisely, by this it is meant that there is an RAE such that, both initially and under a quasi-static loading, the void volume ratio of the domain enclosed by the *convex envelope* of its grains is small enough and the grains are multiply connected with each other. This definition of dense granular materials extends the one of Satake [36] in that it allows a grain to change its contact or no contact status relative to a neighboring grain and excludes "floating" grains which are problematical for any static or quasi-static treatment.

Let N be the total number of the grains belonging to an RAE of the considered granular material. Denoting the set $\{1,2,\ldots,N\}$ by $\mathcal{N}$, each grain of the RAE can be given a unique number $i\in\mathcal{N}$ and identified with the closed domain $\overline{\Omega}^{(i)}$ of $\mathcal{R}^3$ it occupies in the initial configuration. The interior and boundary of $\overline{\Omega}^{(i)}$ are designated by $\Omega^{(i)}$ and $\partial\Omega^{(i)}$. A motion of grain i is defined by a *sufficiently smooth* mapping $\mathbf{y}^{(i)}:\overline{\Omega}^{(i)}\times[0,+\infty[\rightarrow\mathcal{R}^3$, which assigns the position vector $\mathbf{y}^{(i)}$ at any time $t\in[0,+\infty[$ to every $\mathbf{x}^{(i)}\in\overline{\Omega}^{(i)}$:

$$\mathbf{y}^{(i)}=\mathbf{y}^{(i)}\left(\mathbf{x}^{(i)},t\right). \tag{1}$$

The configuration of grain i at time t is then given by $\overline{\omega}^{(i)}=\mathbf{y}^{(i)}\left(\overline{\Omega}^{(i)},t\right)$ with the interior $\omega^{(i)}$ and boundary $\partial\omega^{(i)}$. The displacement is defined by

$$\mathbf{u}^{(i)}=\mathbf{u}^{(i)}\left(\mathbf{x}^{(i)},t\right)=\mathbf{y}^{(i)}\left(\mathbf{x}^{(i)},t\right)-\mathbf{x}^{(i)}, \tag{2}$$

and the (material) velocity by

$$\dot{\mathbf{y}}^{(i)}=\dot{\mathbf{y}}^{(i)}\left(\mathbf{x}^{(i)},t\right)=\partial_t\mathbf{y}^{(i)}\left(\mathbf{x}^{(i)},t\right). \tag{3}$$

The time dependence will be dropped when either it is irrelevant or no risk of confusion is possible.

The unilateral contact law and the notion of an RAE have some important geometrical or kinematical implications. Firstly, the impenetrability condition of the unilateral contact law requires the functions $\mathbf{y}^{(i)}(i\in\mathcal{N})$ to be formally such that

$$\mathbf{y}^{(p)}\left(\Omega^{(p)},t\right)\cap\mathbf{y}^{(q)}\left(\Omega^{(q)},t\right)=\varnothing \text{ for any } p\neq q \text{ and any time } t. \tag{4}$$

Secondly, for an RAE to make sense, it must contain a sufficiently large number of grains and its size must also be sufficiently large with respect to the size of each grain. Letting $d^{(i)}(\mathbf{n})$ be the Feret diameter of $\overline{\Omega}^{(i)}$ in the direction of unit vector $\mathbf{n}$, i.e., the distance between the two parallel planes perpendicular to $\mathbf{n}$ and tangent to $\overline{\Omega}^{(i)}$, and letting $d(\mathbf{n})$ be the corresponding Feret diameter of $\cup_{i=1}^{i=N}\overline{\Omega}^{(i)}$ (Fig. 1), then the latter size requirement can be characterized as

$$d^{(i)}(\mathbf{n})\ll d(\mathbf{n}) \text{ for all } i\in\mathcal{N} \text{ and all } \mathbf{n}. \tag{5}$$

The problem studied in this paper can now be stated as follows:

Given an RAE for a granular material and the corresponding motion functions $\mathbf{y}^{(i)}(i\in\mathcal{N})$, *how to micromechanically define macroscopic measures for the RAE which is assimilated to a representative volume element (RVE) filled with a continuum medium?*

The first difficulty encountered in dealing with this problem is to specify the initial and transformed configurations of the RVE in relation to the ones of the grains of a RAE. One way to circumvent this difficulty is to define the initial and transformed configurations, $\overline{\Omega}$ and $\bar{\omega}$,of the RVE as the *convex combinations* of those of the grains:

$$\overline{\Omega}=\left\{\mathbf{x}\in\mathcal{R}^3|\mathbf{x}=\sum_{i=1}^{i=N}\lambda^{(i)}\mathbf{x}^{(i)},0\leqslant\lambda^{(i)}\leqslant1,\sum_{i=1}^{i=N}\lambda^{(i)}=1,\mathbf{x}^{(i)}\in\cup_{i=1}^{i=N}\overline{\Omega}^{(i)}\right\}, \tag{6}$$

$$\bar{\omega}=\left\{\mathbf{y}\in\mathcal{R}^3|\mathbf{y}=\sum_{i=1}^{i=N}\lambda^{(i)}\mathbf{y}^{(i)},0\leqslant\lambda^{(i)}\leqslant1,\sum_{i=1}^{i=N}\lambda^{(i)}=1,\mathbf{y}^{(i)}\in\cup_{i=1}^{i=N}\bar{\omega}^{(i)}\right\}. \tag{7}$$

The interiors Ω and ω of $\overline{\Omega}$ and $\bar{\omega}$ correspond to the domains enclosed by the convex envelopes $\partial\Omega$ and $\partial\omega$ of the initial and transformed configurations of the grains (Fig. 1). From the standpoint of both experiment and practice, it is meaningful to define the initial and transformed configurations of the RVE associated with an RAE by (6) and (7), since the convex envelope $\partial\Omega$ or $\partial\omega$ is mostly often the actual surface across which the contact forces external to the RAE can be applied. Further, (6) and (7) are particularly suitable for the dense granular materials as defined at the beginning of this section.

We observe that a function that can transform $\overline{\Omega}$ into $\bar{\omega}$ is a priori not defined except over the part of $\cup_{i=1}^{i=N}\overline{\Omega}^{(i)}$ of $\overline{\Omega}$ occupied by the grains. At the same time, such a function is needed to be able to apply the macroscopic kinematical variable definitions of micromechanics [see e.g. 20,21,23,24,32]. Thus, it may appear

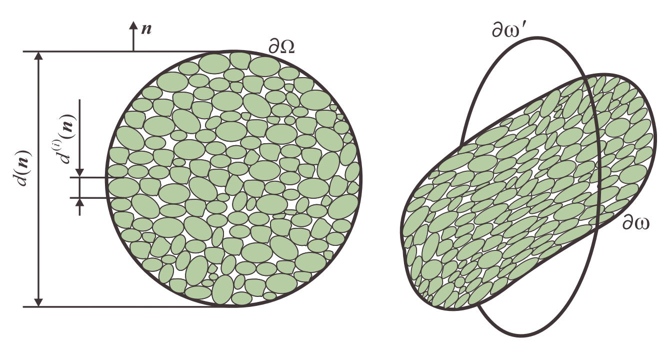

Fig. 1. Convex envelopes of the initial and transformed configurations of an RAE.