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Anisotropic elastic behaviour using the four-dimensional formalism of differential geometry

M. Wang [a], E. Rouhaud [a], A. Roos [a], B. Panicaud [a,*], R. Kerner [b], O. Ameline [a]

[a] Université de Technologie de Troyes (UTT), Institut Charles Delaunay (ICD) – CNRS UMR 6279, Systèmes Mécaniques et d'Ingénierie Simultanée (LASMIS), 12 rue Marie Curie, 10010 Troyes, France

[b] LPTMC, Tour 23-13, Boite Courrier 121, UPMC, 4 Place Jussieu, 75005 Paris, France

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ABSTRACT

This paper discusses covariance and material objectivity in continuum mechanics. The aim is to extend the mathematical framework of constitutive relations to the four-dimensional (4D) formalism of the General Relativity theory. First, it is demonstrated that 4D general linear or non-linear, isotropic or anisotropic relations can be obtained, no matter the reference frame, such as inertial, animated with a rigid body motion and convective or generally curvilinear, and no matter the starting point, for instance a variation of a thermodynamic potential or a direct coupling between stress-like and strain-like tensors, etc. Second, it is then always possible to model anisotropic behaviour, which is not always possible in some of the 3D classical approaches. In order to demonstrate this 4D approach, different elastic relations for different observers have been investigated analytically and numerically.

1. Introduction

The "principle" of material objectivity is an established criterion used in classical 3D mechanics, ensuring frame-indifference of material constitutive models of continuous media [8,23,39]. Such an approach has become essential in solid mechanics where the ever-increasing computing power has made possible numerical models for finite deformations of solids structures, including dissipative effects [22]. When the motion of such structures is considered, a rigid body motion of the matter itself and/or of the frame of reference in which the motion is expressed imposes constraints on the constitutive model. For instance, it has to respect this "principle" of material objectivity.

The theory of General Relativity, by construction, properly deals with space–time transformations (with or without gravity); the covariance principle guarantees proper expression of all physical laws in any kind of reference system, as developed and reviewed by Landau and Lifshitz [20]. As written by Eringen in 1962, indeed: *"Attempts to secure the invariance of the physical laws of motion from the observer have produced one of the great triumphs of twentieth-century physics. (...) Attempts to free the principles of classical mechanics from the motion of an observer were resolved by Einstein in his general theory of relativity..."*. We have proposed to use the covariance principle for application in classical continuum mechanics and demonstrated that this approach is possible and promising. In particular, it can be shown that the four-dimensional (4D) formalism properly distinguishes frame-indifference itself, which is intrinsically the property of a 4D tensor, from the indifference with a superimposed rigid body motion [35]. It can then be concluded that the only appropriate way to define frame-indifference is to apply the covariance principle within the scope of General Relativity theory, and thus to write a 4D constitutive model.

In the present work, anisotropic elastic behaviours are investigated. After recalling the difficulties for finite deformations in 3D, we show how these difficulties can be resolved by the 4D formalism. The 4D tools are presented and a construction method is shown to easily extend isotropic behaviours to anisotropic ones. This generalization has to be done carefully, because this can be done in a partially covariant or in a fully covariant way. It is then demonstrated how the isotropic case can be found by using material symmetries (reflections and rotations). Finally, numerical applications are carried out to illustrate the interest of such an approach.

2. 3D problems and 4D solutions for finite deformations

2.1. 3D problems for finite deformations

A problem occurs when considering the entities that serve to construct 3D constitutive models for finite deformations. Indeed,

* Corresponding author. Tel.: +33 (0)32571806; fax: +33 (0)325715675.
E-mail address: benoit.panicaud@utt.fr (B. Panicaud).

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several constraints need to be considered, for instance the indifference with respect to the superposition of a rigid body motion. This has to be imposed depending on whether solids or fluids are considered [11]. However, in 3D this property coincides with the frame-indifference principle, because any rigid body can be considered as an observer in the 3D sense. Consequently, if this property is not imposed, then the principle of frame-indifference cannot be validated. This major problem on the material objectivity in 3D has provoked many debates [30,31,19,18,32].

Moreover, other difficulties also appear while building 3D constitutive models for finite deformations. For example, dealing with material objectivity, choosing a correct time derivative is not a trivial matter. This has been frequently discussed elsewhere [38,7,5]. Indeed, when computing time derivative without taking into account the local material motion, certain wrong predictions can be made. Furthermore, in 3D, many objective time rates can be defined among countless possibilities [14,13,7]. This has two consequences. First, among the possible time rates, some, even objective, lead to unrealistic behaviour (see for example the Jaumann rate under shear loads oscillating at large deformation [10]). Second, for elastic behaviour, the use of time rates which are not true time derivatives (i.e. not defined as true time differentials) cannot be mathematically integrated like an hyper-elastic model. In other words, such hypo-elastic models do not present a reversible behaviour.

Another question raises when dealing with finite deformations in 3D: how to choose between an Eulerian and a Lagrangean description? This problem is complicated by the possible use of convected coordinates for which correspondence can be established between the components of some Eulerian and Lagrangean tensors [38]. This means that a deeper link exists between the two descriptions.

Finally, the Eulerian description has difficulties when dealing with anisotropic elastic behaviour. Indeed, it is of course always possible to link a 3D elastic strain tensor linearly to a 3D stress tensor using an anisotropic stiffness tensor. However, when constructing models with the representation theory [3], this leads to models, which are not completely Eulerian anymore [38]. This can be related to a kinematic argument: the Eulerian description is not natural for solids, especially the anisotropic ones. This is also of particular interest when considering non-linear behaviours, for instance for applications to bio-materials or elastomers, etc.

2.2. 4D solutions for finite deformations

It is possible to extend the number of dimensions by including time as an extra dimension in order to deal properly with the kinematics of finite deformations. The 4D formalism clearly separates the covariance principle and the indifference with respect to the superposition of a rigid body motion [35,34]. The latter is now a property of the material that can be taken into account (or not), whereas the covariance principle is intrinsically verified through the use of 4D tensors and 4D operators.

The problems concerning time derivatives are now solved by using the only two possible 4D derivatives: the 4D covariant derivative and the 4D Lie derivative. The former is not invariant with respect to the superposition of a rigid body motion, whereas the latter is. Other properties of these derivatives for applications to non-relativistic transformations of continuum mechanics can be found in [35]. These derivatives are also true derivatives that can be properly integrated to obtain reversible elastic behaviours. This will be addressed in a future publication.

The 4D description also encompasses both the Eulerian and Lagrangean descriptions of material transformations (a proof can be found in [34]). The Lagrangean case corresponds to an observer linked to the material (i.e. to the convected frame where the constitutive model can be described), whereas the Eulerian case corresponds to an inertial frame for which the constitutive model has a different expression. Nevertheless, the 4D tensorial relation is intrinsic and corresponds to an expression which is frame-indifferent or covariant, for a given model.

Anisotropic behaviour is now investigated and generalized from isotropic elastic behaviours, for which the construction method will be also briefly presented further.

3. Four-dimensional description of space–time

Differential geometry (also known as Ricci-calculus) proposes a general mathematical context for the description of tensors and the associated algebras. The present section introduces definitions that are necessary for the rest of this work; a detailed presentation may be found in [17,36]. Classical notions of 4D physics are also reviewed in order to introduce specific vocabulary and notations. Further details concerning these subjects are proposed for example in [27,2] where the general concepts are given, while the theory of General Relativity applied to physical fields is presented by Landau and Lifshits [20] and Weinberg [42].

3.1. Coordinates and their transformations

As opposed to classical mechanics point of view, space–time is described by a four-dimensional differentiable manifold. The coordinates of a point within this manifold are parameterized by a set of four real numbers x^μ. This point is called an event and corresponds to a given position and instant of time. The coordinates are denoted with a common index, so that:

$$x^\mu = (x^1, x^2, x^3, x^4) = (x^i, x^4) \tag{1}$$

In this work, the index notation is used and Einstein's summation convention is adopted. Greek indices $\mu, \nu \ldots$ run from 1 to 4 and label a four-dimensional entity. Latin indices i, j run from 1 to 3 and label the spatial part of this entity.

Other sets of coordinates could be indifferently chosen to parameterize the points of the manifold. Consider two possible sets of coordinates noted x^μ and $\tilde{x}^\mu$. The coordinate transformation from x^μ to $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ implies that:

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu \tag{2}$$

The matrix $\Lambda^\mu_\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}$ is the Jacobian matrix of the coordinate transformation and $\left|\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right|$ is the determinant of this Jacobian matrix. Classically, upper indices denote the contravariant components of the tensor, while lower indices denote its covariant components.

3.2. Four-dimensional invariant interval

An invariant interval ds is defined as a generalized length element, such that:

$$ds^2 = (dz^4)^2 - (dz^1)^2 - (dz^2)^2 - (dz^3)^2 = \eta_{\mu\nu} dz^\mu dz^\nu \tag{3}$$

where the coordinates z^μ represent the 4D coordinates of an event and z^i corresponds to the classical 3D cartesian coordinates, as further discussed in Section 4.1. This particular coordinate system is also said to be standard, Galilean or inertial. $\eta_{\mu\nu}$ is the Minkowski metric such that:

$$\eta_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$