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A formulation of a Cosserat-like continuum with multiple scale effects

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ABSTRACT

In this work a generalized continuum formulation is introduced which is based on a theoretical framework of a generalized deformation description proposed by Sansour (1998) [25]. That is the deformation field is composed by macro- and micro-components according to the consideration that the generalized continuum consists of a macro- and micro-continuum. It is demonstrated that by specific definition of the topology of the micro-space this generalized deformation formulation allows for the derivation of a generalized variational principle together with corresponding strain measures and underlying equilibrium equations. The approach makes use of a macroscopic rotation field which is considered to be element of the *Lie* group *SO*(3) and independent of the macroscopic displacement field. In that way the formulation incorporates three additional rotational degrees of freedom and is closely related to the *Cosserat continuum*. In contrast to the conventional *Cosserat continuum* the proposed generalized formulation allows to describe multiple scale effects associated with multiple micro-structural directions, possibly with a different magnitude for each one. The approach considers a geometrically exact description of finite deformation within the macro-continuum, but as a first step linearises the deformation within the micro-continuum. The constitutive law is defined at the microscopic level and the geometrical specification of the micro-continuum is the only material input which goes beyond that needed in a classical description.

Various meshfree computations demonstrate that this model is able to address fundamental physical phenomena which are related to the underlying micro-structure of the material, in particular scale-effects and oriented material behaviour. Clear differences are revealed between a classical and the non-classical formulation.

1. Introduction

Within the last decades the development of generalized continuum models have attracted much interest. The range was spanned from theories designed for very specific physical phenomena to frameworks which had unifying ambitions and were basis for various approaches which were derived from them. Usually, generalized continua are linked with continuum mechanics theories accounting for degrees of freedom additional to the classical displacement field. Considering only extra rotational degrees of freedom Eringen and Kafadar [10] used the term *micropolar continuum*. The origin of this kind of continuum, however, traces back as far as to the Cosserat brothers [8] or even Voigt [38]. The common idea is the micro-structure motivated deformation description and provision for an internal length associated with the characteristic size of the material's underlying micro-structure. Accordingly, isotropic micropolar elastic solids account for at least one material constant more. Furthermore, each material point is equipped with a set of directors, the deformation of which gives rise to higher-order strains and stresses. Within the context of micropolar continua these directors only perform in terms of rigid body rotations so that besides the conventional displacement field the deformation is also characterized by an independent rotation field. Accordingly, upon application of an external stimulus we find force as well as couple stress fields (torque per unit area). Experimental validation of micropolar continuum models are still scarce, e.g. for porous material such as polymeric foam and human bone [14], circular-cell polycarbonate honeycomb structures [20], granular material [37] or plastic hardening of thin copper wires [12].

Initially, the focus of micropolar continuum theories was drawn to the elastic case and often, only the small strain case was envisaged. Lippmann [17] and Besdo [4] provided the first attempts to extend the geometrically linear *Cosserat continuum* to the realm of plasticity. It was subsequently discovered that in case of localized deformation the incorporation of an intrinsic length scale provided remedy concerning the ellipticity loss of the governing equations and the pathological mesh dependency of finite element

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solutions [22,21,5]. Non-linear continuum theories considering finite deformation and rotation were proposed by various authors [3,27,31,33,26,24,35,23].

In contrast to multi-scale computational homogenization techniques, e.g. [34,19,36,13,16] where micro-structural information is condensed from a separately discretized and solved microscopical problem, the attractivity of generalized continua is due to the ability to describe orientation and size of the material constituting particles solely in mathematical terms and in a completely natural way. In particular, micropolar continua are predestined to predict physical phenomena which are related to the micro-structure. Examples are the modelling of size effects in crystal plasticity as found in metal alloys [11,7], shear banding in granular material [39], accounting for interparticle forces and the density variation of granular media [41] or non-local as well as surface related effects in electro-elasticity Shen and Hu [30].

The micropolar formulation proposed in this article is derived from a unified generalized continuum framework [25] exploring the specific case of rotations as extra degrees of freedom and examining the consequences of different choices of micro-structural orientation and intrinsic length with regards to elastic scale effects and possible anisotropy resulting from these specific choices. Similar to multi-scale computational homogenization techniques, this framework also involves micro-structural homogenization to average the non-local material response, i.e. (i) the constitutive law is only defined on micro-structural level, (ii) macro- and micro-deformations as well as relative deformations between macro- and micro-spaces are included. A significant difference, however, is (iii) the incorporation of geometrical and physical micro-structural information. Here, the mathematical concept of a fibre bundle is considered to construct a generalized space. The involved spaces, i.e. generalized, macro- and micro-space, as well as their tangent spaces are linked with each other via projection maps. In particular, the tangent of the projection map defines the geometry of the extra space and so the metric which is to be used in the evaluation of the integrals over the micro-space. The concept is rich in its structure, e.g. it allows for the micro-space to be equipped with further geometric structures such as curvature, should this be deemed relevant from a physical point of view. Here, the trivial identity map is chosen, as this is sufficient to satisfy the primary aim of this work to predict elastic scale effects as well as oriented material behaviour. Also, the difference in the number of material parameters in comparison to a classical formulation is kept to a minimum and is confined to purely geometric ones which describe the micro-continuum.

Finally, another aspect of this approach is worthwhile to be highlighted. The micropolar continuum is modelled in a very general manner opening the door for inelastic formulations considering the geometrically exact description of finite deformations within the macro-continuum but accommodating also higher-order deformations within the micro-continuum.

Now, the plan of this work is as follows: In Section 2 the theory of the generalized continuum is outlined followed by Section 3 where the details of a micropolar continuum theory based on the generalized continuum framework are given. Subsequently, in Section 4 a micropolar variational formulation is derived. The suitability of this micropolar approach to model size-scale effects as well as oriented material behaviour is illustrated in Section 5 by various applications within the domain of hyperelasticity.

2. Deformation and strain

The basic idea is that a generalized continuum $\mathcal{G}$ can be assumed to inherit the mathematical structure of a fibre bundle (see e.g. [6]. In the simplest case, this is the Cartesian product of a macro-space $\mathcal{B} \subset \mathbb{E}(3)$ and a micro space $\mathcal{S}$ which we write as

$$\mathcal{G} := \mathcal{B} \times \mathcal{S}. \tag{1}$$

This definition assumes an additive structure of $\mathcal{G}$ which implies that the integration over the macro- and the micro-continuum can be performed separately. The macro-space $\mathcal{B}$ is parametrized by the curvilinear coordinates ϑ^i, $i = 1, 2, 3$ and the micro-space or micro-continuum $\mathcal{S}$ by the curvilinear coordinates ζ^α. Here, and in what follows, Greek indices take the values 1, ... or n. The dimension of $\mathcal{S}$ denoted by n is arbitrary, but finite. Furthermore, we want to exclude that the dimension and topology of the micro-space is dependent on ϑ^i. As illustrated in Fig. 1 each material point $\widetilde{\mathbf{X}} \in \mathcal{G}$ is related to its spatial placement $\tilde{\mathbf{x}} \in \mathcal{G}_t$ at time $t \in \mathbb{R}$ by the mapping

$$\tilde{\boldsymbol{\varphi}}(t) : \mathcal{G} \to \mathcal{G}_t. \tag{2}$$

For convenience but without loss of generality we identify $\mathcal{G}$ with the un-deformed reference configuration at a fixed time t_0 in what follows. The generalized space can be projected to the macro-space in its reference and its current configuration by

$$\pi_0(\widetilde{\mathbf{X}}) = \mathbf{X} \text{ and } \pi_t(\tilde{\mathbf{x}}) = \mathbf{x}, \tag{3}$$

respectively, where π_0 as well as π_t represent projection maps, and $\mathbf{X} \in \mathcal{B}$ and $\mathbf{x} \in \mathcal{B}_t$. The tangent space $\mathcal{TG}$ in the reference configuration is defined by the pair $(\widetilde{\mathbf{G}}_i \times \mathbf{I}_\alpha)$ given by

$$\widetilde{\mathbf{G}}_i = \frac{\partial \widetilde{\mathbf{X}}}{\partial \vartheta^i} \text{ and } \mathbf{I}_\alpha = \frac{\partial \widetilde{\mathbf{X}}}{\partial \zeta^\alpha}, \tag{4}$$

where the corresponding dual contra-variant vectors are denoted by $\widetilde{\mathbf{G}}^i$ and $\mathbf{I}_\alpha$, respectively. A corresponding tangent space in the current configuration $\mathcal{TG}_t$ is spanned by the pair $(\tilde{\mathbf{g}}_i \times \mathbf{i}_\alpha)$ given by

$$\tilde{\mathbf{g}}_i = \frac{\partial \tilde{\mathbf{x}}}{\partial \vartheta^i} \qquad \text{and} \qquad \mathbf{i}_\alpha = \frac{\partial \tilde{\mathbf{x}}}{\partial \zeta^\alpha}. \tag{5}$$

The generalized tangent space can also be projected to its corresponding macro-space by

$$\pi_0^*(\widetilde{\mathbf{G}_i}) = \mathbf{G}_i \text{ and } \pi_t^*(\tilde{\mathbf{g}}_i) = \mathbf{g}_i, \tag{6}$$

respectively, where the tangent vectors $\mathbf{I}_\alpha$ are assumed to be constant throughout $\mathcal{S}$ for simplicity. Note that the definition of a projection map is not trivial. The tangent of the projection map defines the geometry of the extra space and so the metric which is to be used to evaluate the integral over the generalized space. The concept is rich in its structure.

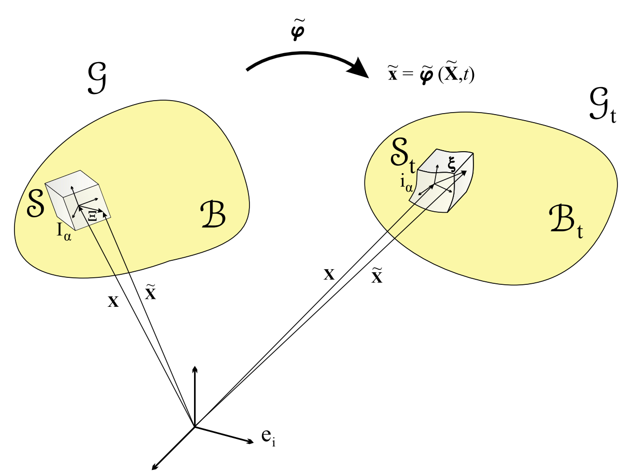

Fig. 1. Configuration spaces.