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Variational formulation and finite element analysis for nonlocal elastic nanobeams and nanoplates

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ABSTRACT

In the present work, finite element formulations for nonlocal elastic (i) Euler–Bernoulli beam and (ii) Kirchhoff plate have been reported. Nonlocal differential elasticity theory is considered. Galerkin finite element technique has been employed. For both nanobeams and nanoplates weak forms of governing equations are derived and energy functionals are obtained. Present finite element results for bending, vibration and buckling for nonlocal beam with four classical boundary conditions are computed. These results are in good agreement with those reported in the literature. Further, bending, vibration and buckling analyses are carried out for stepped nanobeam. Furthermore, using present finite element bending, vibration and buckling analyses for nonlocal nanoplate are carried out. Present formulation will be useful for structural analyses of nanostructures with complex geometry, material property, loading and boundary conditions.

1. Introduction

Due to outstanding physical, chemical, mechanical and electronic properties [1–4], nano-sized structures [5–8] have attracted a great deal of attention in scientific community. As conduction of experiments in nano-level are difficult to control and theoretical atomistic models are computationally intensive for relatively large scale nanostructures, the continuum and semi-continuum [9–11] models have been proven to be important tools in the study of the nanostructures. Considering the continuum models, the use of traditional elasticity theory may lead to erroneous results as continuum assumption may not hold valid in the small scales. This fact triggered development of various micro-continuum theories such as couple stress theory [12], micro-morphic theory [13], strain gradient elasticity theory [14] and nonlocal elasticity theory [15]. Among these theories, nonlocal elasticity theory has been widely applied to various problems of physics including lattice dispersion of elastic waves and dislocation mechanics [16].

Chen et al. [17] proved that nonlocal elasticity theory is consistent with the molecular dynamics. This has made the nonlocal elasticity theory an efficient alternate to atomistic methods. Initially Peddieson et al. [18] applied present nonlocal elasticity theory and studied flexural behavior of one dimensional nanostructures. Since then a large number of research activities using nonlocal elasticity theory have taken place. These include analyses of nanobeams [19–26], nanoplates [27–32] and nanoshells [33–35]. In the analyses of nanostructures, nonlocal differential elasticity (or nonlocal stress gradient elasticity) has gained more popularity among researchers as compared to the nonlocal integral elasticity [33] due to its simplicity.

So far two methodologies have been used extensively for solving the governing differential equations arising in structural analysis of nonlocal elastic nanostructures. These are Navier's method [27,28,32] and Differential Quadrature Method (DQM) [20,21,29,30]. However it is well known that finite element method in contrast to these methods can effectively handle more complex geometry, material property, boundary and/or loading conditions. Construction of variational principles in the framework of nonlocal integral elasticity has been reported by Polizzotto [36]. Pisano et al. [37,38] reported the finite element procedure for nonlocal integral elasticity. Recently Adali [39,40] and Shakouri et al. [41] have devised traditional variational methods based on nonlocal differential elasticity for the analysis of carbon nanotube. However to the authors' best knowledge, the finite element analysis of nonlocal elastic structures with nonlocal differential elasticity approach is not available in the open literature. In the present work, finite element formulation for nonlocal differential elasticity approach of Euler–Bernoulli beam theory and classical plate theory have been reported. Expressions for quadratic functionals have also been given.

2. Review of governing equations

2.1. Nonlocal beam

Formulation for bending, vibration and buckling of nonlocal Euler–Bernoulli beam has been reported by Reddy [42]. To write the

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Nomenclature

a, b	length and width of the plate element
$[B^b]$, $[B^p]$	buckling stiffness matrix for beam and plate respectively
D_{11}, D_{12}, D_{22}, D_{66}	bending rigidities of the plate
Eb	Young's modulus of the beam material
$E1$, $E2$	Young's moduli of the plate material
Fb, Fp	functional for beam and plate respectively
$G12$	shear modulus of plate the plate material
h	thickness of the plate
$[K^b]$, $[K^p]$	stiffness matrix for beam and plate respectively
l	length of the beam element
m_0^b, m_0^p	mass moment of inertia for beam and plate respectively
Mb	moment resultant for the beam
$M_1^{xx}, M_1^{yy}, M_1^{xy}$	moment resultants for the plate
(n, s)	local tangential and normal coordinate at the boundary
N_i^b, N_i^p	interpolation function for beam and plate respectively
$N_0^{xx}, N_0^{yy}, N_0^{xy}$	in-plane stress resultants for plate
$[M^b]$, $[M^p]$	mass matrix for beam and plate respectively
M_n, M_{ns}, V_n	boundary stress resultants
P	axial load for the beam
$\widehat{P}_{cr}^b, \widehat{P}_{cr}^p$	critical buckling loads for beam and plate respectively
q^b, q^p	transverse distributed load for beam and plate respectively
Q_x, Q_y	shear stress resultants for the plate
SS, CL, FR	simply-supported, clamped and free boundary conditions respectively
u_i^b, u_i^p	degree of freedom for beam and plate respectively
$[U^b]$, $[U^p]$	degree of freedom vector for beam and plate respectively
Vb	shear stress resultant for the beam
$\hat{w}^b, \hat{w}^p$	non-dimensional maximum deflection of beam and plate respectively
w^b (or W^b), w^p (or W^p)	deflection of beam and plate respectively
χ^b, χ^p	weight function for beam and plate respectively
ε_{xx}	axial strain tensor for the beam
λ^b, λ^p	ratio of actual buckling load and applied in-plane loads for beam and plate respectively
μ	nonlocal parameter
ν_{12}, ν_{21}	Poisson's ratios of plate
ρ^b, ρ^p	density of the beam material and plate material respectively
ω^b, ω^p	frequency of beam and plate respectively
$\hat{\omega}^b, \hat{\omega}^p$	non-dimensional natural frequency of beam and plate respectively
σ_{xx}	axial stress tensor for the beam
∇^2	Laplacian operator in two dimensional cartesian coordinate system

governing equations, we assume the follow reference coordinate system: x, y and z axes are taken along the length, width and thickness of the beam, respectively. Using the principle of virtual work following equilibrium equations are obtained:

$$\frac{\partial^2 M^b}{\partial x^2} + q^b - P\frac{\partial^2 w^b}{\partial x^2} - m_0^b\frac{\partial^2 w^b}{\partial t^2} = 0 \tag{1}$$

$$V^b - \frac{\partial M^b}{\partial x} + P\frac{\partial w^b}{\partial x} = 0 \tag{2}$$

The constitutive relation according to nonlocal differential elasticity takes the following form for the beam:

$$\sigma_{xx} - \mu\frac{\partial^2\sigma_{xx}}{\partial x^2} = E^b\varepsilon_{xx} \tag{3}$$

The stress resultants can be written in terms of displacements as follows:

$$M^b = -(E^bI^b - \mu P)\frac{\partial^2 w^b}{\partial x^2} - \mu q^b + \mu m_0^b\frac{\partial^2 w^b}{\partial t^2} \tag{4}$$

$$V^b = -(E^bI^b - \mu P)\frac{\partial^3 w^b}{\partial x^3} - \mu\frac{\partial q^b}{\partial x} + \mu m_0^b\frac{\partial^3 w^b}{\partial x\partial t^2} - P\frac{\partial w^b}{\partial x} \tag{5}$$

Putting Eq. (4) into Eq. (1) and assuming, $w^b(x,t) = W^b(x)e^{i\omega^b t}$ we get the following governing equation in terms of displacement:

$$E^bI^b\frac{\partial^4 W^b}{\partial x^4} - \left(1 - \mu\frac{\partial^2}{\partial x^2}\right)\left\{q^b(x) - P\frac{\partial^2 W^b}{\partial x^2} + m_0^b\omega^{b2}W^b - m_0^b\omega^{b2}\frac{\partial^2 W^b}{\partial x^2}\right\} = 0 \tag{6}$$

It can be noted that, by putting $\mu = 0$, in Eq. (6), governing equation for a local beam can be retrieved.

2.2. Nonlocal plate

Formulation for bending, vibration and buckling of nonlocal plate based on classical plate theory has been reported by Pradhan and Phadikar [27,28,46]. Following coordinate system has been chosen to write the equations. Origin is chosen at one corner of the plate. x, y and z coordinate axes are taken along the length, width and thickness of the plate, respectively. Following equilibrium equations is obtained using the principle of virtual work:

$$\frac{\partial^2 M_1^{xx}}{\partial x^2} + 2\frac{\partial^2 M_1^{xy}}{\partial x\partial y} + \frac{\partial^2 M_1^{yy}}{\partial y^2} + q^p + \frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w^p}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w^p}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w^p}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w^p}{\partial x}\right) - m_0^p\frac{\partial^2 w^p}{\partial t^2} = 0 \tag{7}$$

$$Q_x - \frac{\partial M_1^{xx}}{\partial x} - \frac{\partial M_1^{xy}}{\partial y} - N_0^{xx}\frac{\partial w^p}{\partial x} - N_0^{xy}\frac{\partial w^p}{\partial y} = 0 \tag{8}$$

$$Q_y - \frac{\partial M_1^{xy}}{\partial x} - \frac{\partial M_1^{yy}}{\partial y} - N_0^{yy}\frac{\partial w^p}{\partial y} - N_0^{xy}\frac{\partial w^p}{\partial x} = 0 \tag{9}$$

Using the nonlocal constitutive relation, stress resultants are written in terms of displacements as follows:

$$M_1^{xx} - \mu\nabla^2 M_1^{xx} = -D_{11}\frac{\partial^2 w^p}{\partial x^2} - D_{12}\frac{\partial^2 w^p}{\partial y^2} \tag{10}$$

$$M_1^{yy} - \mu\nabla^2 M_1^{yy} = -D_{12}\frac{\partial^2 w^p}{\partial x^2} - D_{22}\frac{\partial^2 w^p}{\partial y^2} \tag{11}$$

$$M_1^{xy} - \mu\nabla^2 M_1^{xy} = -2D_{66}\frac{\partial^2 w^p}{\partial x\partial y} \tag{12}$$

Here Dij are various bending rigidities defined as

$$D_{11} = \frac{E_1h^3}{12(1-\nu_{12}\nu_{21})}, \quad D_{12} = \frac{\nu_{12}E_2h^3}{12(1-\nu_{12}\nu_{21})}, \quad D_{22} = \frac{E_2h^3}{12(1-\nu_{12}\nu_{21})}, \quad D_{66} = \frac{G_{12}h^3}{12} \tag{13}$$

Using Eqs. (10)–(12) and Eq. (7) and assuming $w^p(x,t) = W^p(x)e^{i\omega^p t}$ we get the following governing equation in terms of displacements: