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An alternative numerical solution of thick-walled cylinders and spheres made of functionally graded materials

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ABSTRACT

In the problem of thick-walled cylinders of functionally graded materials (FGMs), this paper chooses the displacement function as unknown in the governing equation. The Young's modulus is arbitrary and constant Poisson's ratio is assumed. Two fundamental solutions are defined from a numerical solution under two particular initial boundary conditions. In addition, the transmission matrix **M** is defined for the single layer case. The matrix relates the values of radical stress and displacement at the initial point to those at the end point of the layer. By using the matrix, the boundary value problem can be solved as well. After considering the continuation condition along the layers, the method of transmission matrix can also be used to the multiply-layered cylinder. The method of transmission matrix can also be used to solve the problem of the thick-walled sphere. A lot of numerical examples are presented.

1. Introduction

Functionally graded materials (FGMs) possess some particular advantages. For example, the materials can be used in a high temperature environment. Therefore, analysis for components made of FGMs got much attention by many researchers. The problems for FGM vessels under higher inner pressure is a particular problem in this field. The usage of FGMs in pressurized vessels can considerably change the stress distribution along the section. However, FGMs are inhomogeneous material. Therefore, the liner theory of elasticity is no longer useful for the study of FGM component.

For the cylinder of homogenous property, the problem has been solved completely [1,2]. Many papers studied the problem of FGM cylinder [3–11]. In order to get an analytical solution for the problem, most researchers assumed the Young's elastic modulus in the form of power law or exponential function. In Ref. [11], authors proposed a solution for an arbitrary response of the Young's modulus. In the suggested method, one still needs to solve a Fredholm integral equation numerically. Using the stress function as unknown, the boundary value problem for a FGM cylinder was solved by a superposition of two initial boundary value problems [10]. Plastic limit angular velocity of rotating hollow cylinders made of the von Mises materials with nonlinear isotropic hardening is investigated numerically and analytically [12].

From above-mentioned results we see that it is necessary to develop some numerical procedures for the case of an arbitrary Young's modulus. In this paper, similar studies were devoted to the pressurized sphere made of FGMs [13,14].

This paper starts on a study of single layer cylinder of FGM. The Young's modulus is assumed to be an arbitrary function with respect to "r", and the Poisson's ratio takes a constant value. A transmission matrix **M** is defined for the single layer case ($a \leqslant r \leqslant b$). The matrix relates the values of radical stress and displacement at the initial point ($r = a$) to those at the end point ($r = b$). The matrix is evaluated on a numerical solution of two fundamental solutions. The matrix is an inherent property of the cylinder, which only depends on the geometry and elastic response of the cylinder and has no relation with imposed boundary conditions. The boundary value problem can be easily solved by using the transmission matrix. In addition, the method of transmission matrix is also a powerful tool to solve the problem of multiply-layered cylinder of FGM. Similarly, the transmission matrix method can also be used to the problem of pressurized thick-walled sphere.

In literature, there are two ways for the choice of the unknown function in the ordinary differential equation of FGM cylinder. Among them, one is the stress function and the other is the displacement function.

It is valuable to develop different methods to solve the same or similar problems for FGM cylinder. This can provide a deeper understanding for the problem. In literatures, some researchers devoted their effort to obtain a closed form solution for FGM cylinder. Therefore, those researchers might choose the better one between two methods, or the stress function and the displacement function methods, to reach their goal. In this paper, a numerical integration method based on the displacement function is suggested. This

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adoption provides a variety of solution. A lot of numerical examples are carried out in the present study, and the influences of the elastic response to the stress distribution on the section are studied in detail.

2. Boundary value problem for single layer cylinder of FGMs with general elastic response

In the history of elasticity there are two ways for the solution of the boundary value problem [1]. Among them, one is based on the stress function as unknown function, and the other is based on the displacement. A similar situation happens in the elastic analysis of the cylinder of FGMs with general elastic response.

For solving the boundary value problem of a thick-walled cylinder made of FGMs with general elastic response, an ordinary differential equation is derived below in which the displacement is taken as unknown function. The concept of transmission matrix is introduced, which is a powerful tool to solve the problem. Based on the transmission matrix, the problem for multiply-layered cylinder can also be solved. Many numerical examples and computed results are provided.

2.1. Formulation of ordinary differential equation based on the displacement function for a thick-walled cylinder made of FGMs

A long cylinder with inner radius "a" and outer radius "b" is investigated (Fig. 1). The cylinder is assumed under some traction or displacement boundary value conditions at the inner boundary ($r = a$) and outer boundary ($r = b$). The problem can be studied in the polar coordinates (r, θ).

The displacement in the r-direction is denoted by "u". Two strain components can be expressed as [1]

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \frac{u}{r} \qquad (\text{with } u = \varepsilon_\theta r) \tag{1}$$

In the present study, the Young's modulus $E(r)$ is an arbitrary function, and the Poisson's ratio takes a constant value $\nu = 0.3$. In the plane strain case, the stress–strain relation takes the form

$$\varepsilon_r = \frac{1-\nu^2}{E(r)}\left(\sigma_r - \frac{\nu}{1-\nu}\sigma_\theta\right) \tag{2}$$

$$\varepsilon_\theta = \frac{1-\nu^2}{E(r)}\left(\sigma_\theta - \frac{\nu}{1-\nu}\sigma_r\right) \tag{3}$$

From Eqs. (1)–(3), the stress components can be expressed as

$$\sigma_r = \frac{E(r)(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{du}{dr} + \frac{\nu}{1-\nu}\frac{u}{r}\right) \tag{4}$$

$$\sigma_\theta = \frac{E(r)(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{u}{r} + \frac{\nu}{1-\nu}\frac{du}{dr}\right) \tag{5}$$

In the symmetrical deformation case, the equilibrium equation for the stress components σ_r and σ_θ takes the form

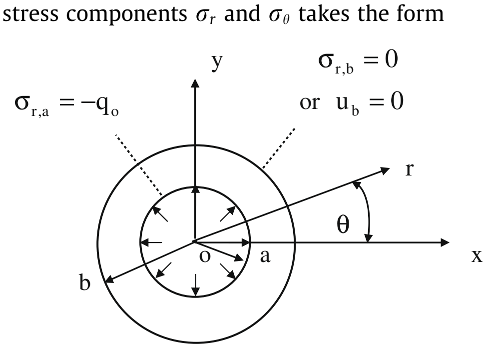

Fig. 1. Two typical boundary conditions: (a) $\sigma_{r,a} = -q_o$ at $r = a, \sigma_{r,b} = 0$ at $r = b$, (b) $\sigma_{r,a} = -q_o$ at $r = a,\ u_b = 0$ at $r = b$.

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{6}$$

Substituting Eqs. (4) and (5) into Eq. (6) yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + \left(\frac{du}{dr} + \frac{\nu}{1-\nu}\frac{u}{r}\right)\frac{1}{E(r)}\frac{dE(r)}{dr} = 0 \ (\text{or } \Lambda(u(r)) = 0) \tag{7}$$

The problem is studied within the range $a \leqslant r \leqslant b$. In the analysis, the following notations are used

$$\sigma_{r,a} = \sigma_r|_{r=a}, \quad u_a = u|_{r=a} \tag{8}$$

$$\sigma_{r,b} = \sigma_r|_{r=b}, \quad u_b = u|_{r=b} \tag{9}$$

In Eqs. (8) and (9), $\sigma_{r,a},\ \sigma_{r,b}$ denote the stress σ_r at $r = a$ or $r = b$, and $u_a,\ u_b$ denote the displacement "u" at $r = a$ or $r = b$, respectively.

For a single layer cylinder case, there are four possibilities to formulate the boundary problems. They are as follows

$$\sigma_{r,a} = f_1, \quad \sigma_{r,b} = f_2 \tag{10a}$$

$$\sigma_{r,a} = f_1, \quad u_b = g_2 \tag{10b}$$

$$u_a = g_1, \quad \sigma_{r,b} = f_2 \tag{10c}$$

$$u_a = g_1, \quad u_b = g_2 \tag{10d}$$

where $f_1,\ f_2,\ g_1$ and g_2 are values given beforehand.

Clearly, once a solution for the displacement "u" is obtained, from Eqs. (1)–(5), the components $\varepsilon_r,\ \varepsilon_\vartheta,\ \sigma_r$ and σ_θ can be obtained accordingly.

2.2. Formulation of the transmission matrix in the case of single layer cylinder

Physically, if the two initial values for $\sigma_{r,a}$ and u_a are assumed at $r = a$, we have definite values for $\sigma_{r,b}$ and u_b at $r = b$. The relation between $(\sigma_{r,a},\ u_a)$ and $(\sigma_{r,b},\ u_b)$ can be expressed through a matrix, which is called the transmission matrix **M** hereafter. The introduced matrix **M** is used not only to the single layer case but also to the multiply-layered case. In the following analysis, a technique for finding the matrix **M** will be introduced.

In the analysis, two fundamental functions $s_1(r)$ and $s_2(r)$ are introduced, which are defined by

$$\Lambda(s_1(r)) = 0,\ s_1|_{r=a} = 1,\ \left.\frac{ds_1}{dr}\right|_{r=a} = 0, \quad (\text{the first fundamental function}) \tag{11}$$

$$\Lambda(s_2(r)) = 0,\ s_2|_{r=a} = 0,\ \left.\frac{ds_2}{dr}\right|_{r=a} = 1, \quad (\text{the second fundamental function}) \tag{12}$$

In Eqs. (11) and (12), Λ is a differential operator acting upon the displacement function, which was indicated in detail by Eq. (7).

It is assumed that the studied solution for $u(r)$ can be expressed as

$$u(r) = c_1 s_1(r) + c_2 s_2(r) \tag{13}$$

From Eqs. (4), (11), (12) and (13), we have

$$\begin{aligned}\sigma_{r,a} &= c_1\frac{E(a)(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{\nu}{1-\nu}\frac{s_1}{r} + \frac{ds_1}{dr}\right)_{r=a} + c_2\frac{E(a)(1-\nu)}{(1+\nu)(1-2\nu)} \\ &\quad \times \left(\frac{\nu}{1-\nu}\frac{s_2}{r} + \frac{ds_2}{dr}\right)_{r=a} \\ &= \frac{E(a)(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{\nu}{1-\nu}\frac{c_1}{a} + c_2\right)\end{aligned} \tag{14}$$

$$u_a = c_1 s_1(r)|_{r=a} + c_2 s_2(r)|_{r=a} = c_1 \tag{15}$$