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Asymptotic homogenisation in linear elasticity. Part I: Mathematical formulation and finite element modelling

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ABSTRACT

The asymptotic expansion homogenisation method is an excellent methodology to model physical phenomena on media with periodic microstructure and a useful technique to study the mechanical behaviour of structural components built with composite materials. In the first part of this work the authors present a detailed form of the mathematical formulation of the asymptotic expansion homogenisation for linear elasticity problems, as well the explicit mathematical equations that characterise the microstructural stress and strain fields associated with a given macrostructural equilibrium state – the localisation procedure. From this mathematical basis, the authors also present the numerical equations resulting from the finite element modelling of the asymptotic expansion homogenisation method in linear elasticity.

1. Introduction

The numerical simulation of the mechanical behaviour of complex microstructure composite materials often leads to the need for significant resources in terms of memory and CPU time. In this context, numerical models used to predict the behaviour of these materials are developed. One of these methods is the asymptotic expansion homogenisation (AEH). Applying the AEH, overall material properties can be derived from the mechanical behaviour of selected microscale representative volumes (also known as representative unit-cells, RUC) with specific periodic boundary conditions.

The homogenisation method equations for cellular media have been previously presented, in a weak form derivation, by Guedes and Kikuchi [1]. Although this derivation is suitable for the finite element method formulation, the use of the strong form allows a better understanding of the mathematical and physical multiscale relations of the method. With this in mind, the authors present a detailed strong form derivation of the asymptotic expansion homogenisation for linear elasticity problems. In addition, some of the main aspects of the higher-order displacement correctors and localisation procedure, as well as the numerical equations resulting from the finite element modelling of the method, are also discussed.

2. Asymptotic expansion homogenisation in linear elasticity

The detailed numerical modelling of the mechanical behaviour of composite material structures often involves high computational costs. In this scope, the use of homogenisation methodologies can lead to significant benefits. These techniques allow the substitution of a heterogeneous medium for an equivalent homogenous medium, therefore allowing the use of macrostructural behaviour laws obtained from microstructural information.

On the other hand, composite materials typically have heterogeneities with characteristic dimensions much smaller than the dimensions of the structural component itself. If the distribution of the heterogeneities is roughly periodical, it can usually be approximated by the periodical repetition of a unit-cell, representative of the microstructural details of the composite material.

With this in mind, the asymptotic expansion homogenisation method is an excellent methodology to model physical phenomena

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on media with periodic microstructure, as well as a useful technique to study the mechanical behaviour of structural components built with composite materials. The main numerical advantages of this method are (i) the fact that it allows a significant reduction of the number of degrees of freedom and (ii) the capability to find the stress and strain microstructural fields associated with a given macrostructural equilibrium state. In fact, unlike other common homogenisation methods, the AEH leads to explicit mathematical equations to characterise those fields. This process is often called localisation.

The concept of homogenisation applied to physical properties goes back to the nineteenth century [2–4]. However, the first published work on mathematical homogenisation theory dates to the end of the 1960s [5–7]. Since then, several homogenisation methods have been proposed, from which the AEH stands out as one of the most important [8,9]. This method allows the calculation of average (homogenised) values for the mechanical properties of a given composite material. However, the AEH cannot be used with non-periodical microstructures. In this case, other homogenisation methodologies are better suited, such as the G-convergence [5] – for symmetric non-periodic problems –, H-convergence [10] – for non-symmetric non-periodic problems – and Γ-convergence [11] – for problems that allow a variational formulation. Even so, these methodologies do not lead to the homogenised properties the composite material, but only to estimates for its upper and lower bounds.

In this section the authors present the main features associated to the application of the AEH methodology to the linear elasticity problem. Some notions about higher-order terms of the asymptotic expansion of the displacement field are also presented and the formulation for the localisation problem is shown.

2.1. Differential formulation of the elasticity problem

Consider a heterogeneous linear elastic material associated to a material body Ω. The microstructure of Ω is made of a spatially periodic distribution of a representative unit-cell associated to a region Y, as illustrated in Fig. 1. Most heterogeneous materials with periodic microstructures have a very small relation ϵ ($\epsilon \ll 1$) between the micro- and macroscale characteristic dimensions. The mechanical loading of such materials creates periodic oscillations on the resulting displacement, stress or strain fields. These oscillations are a consequence of the periodicity of the microstructural heterogeneities and are evident in a neighbouring region of dimension ϵ of any point of Ω. In this context, the existence of two different scales **x** and **y**, each associated respectively to the material behaviour phenomena on the macroscale Ω and microscale Y levels (see Fig. 1), is often assumed. Thus, the variables associated to the referred fields become functionally dependent on both **x** and **y** systems, where

$$\mathbf{y} = \mathbf{x}/\epsilon. \tag{1}$$

The functional dependence on **y** is periodical in Y. This property is usually called Y-periodicity. In terms of elastic properties, the Y-periodicity of the microstructural heterogeneities reflects itself on the fact that the elasticity tensor **D** is Y-periodic in **y**. In contrast, the material homogeneity at the macroscale level results from the fact that the elasticity tensor components do not depend on the macroscale system of coordinates, **x**. Therefore, the components of the elasticity tensor are

$$D_{ijkl} = D_{ijkl}(\mathbf{y}). \tag{2}$$

However, on the macroscale system of coordinates, **x**, the microstructural heterogeneity appears over periods ϵ^{-1} times smaller than the characteristic dimension of the domain Y. According to Eq. 1, this is denoted by

$$D^{\epsilon}_{ijkl}(\mathbf{x}) = D_{ijkl}(\mathbf{x}/\epsilon), \tag{3}$$

where the superindex ϵ stands for the ϵ Y-periodicity of **D** in the system of coordinates **x**, thus depending indirectly on **x**. Assuming infinitesimal strains associated to a static equilibrium state and considering Einstein's convention for tensor notation, the linear-elasticity problem can be described using equilibrium equations, linearised strain–displacement relations and constitutive laws that may be written as [12]

$$\frac{\partial \sigma^{\epsilon}_{ij}}{\partial x^{\epsilon}_j} + f_i = 0 \qquad \text{in } \Omega, \tag{4}$$

$$\varepsilon^{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u^{\epsilon}_i}{\partial x^{\epsilon}_j} + \frac{\partial u^{\epsilon}_j}{\partial x^{\epsilon}_i}\right) \qquad \text{in } \Omega \text{ and} \tag{5}$$

$$\sigma^{\epsilon}_{ij} = D^{\epsilon}_{ijkl}\varepsilon^{\epsilon}_{kl} \quad \text{in } \Omega, \tag{6}$$

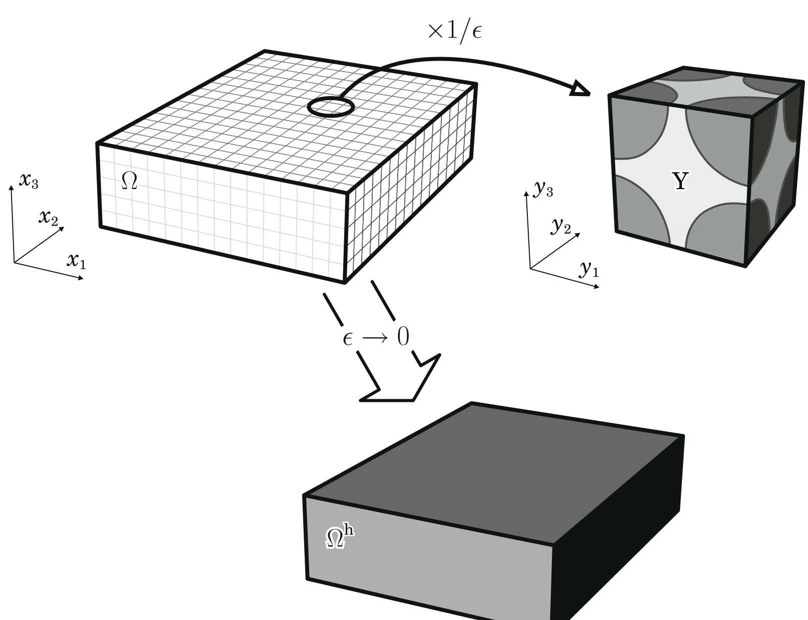

Fig. 1. Schematic representation of the heterogeneous elastic material Ω and the unit-cell Y, representative of the microscale, used in the asymptotic expansion homogenisation method, which results, with $\epsilon \to 0$, in the homogenous material Ω^{h}.