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Realizable algebraic Reynolds stress closure

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ABSTRACT

The normalized Reynolds (NR-) stress is a symmetric, non-negative, dyadic-valued operator. An analysis of the hydrodynamic equation governing velocity fluctuations of a constant property Newtonian fluid shows that this operator is related to a prestress operator that is also symmetric and non-negative. The prestress operator accounts for local spatial changes in the fluctuating pressure and in the fluctuating instantaneous Reynolds stress. The Cayley–Hamilton theorem from linear algebra is used to complete the closure with a non-negative mapping of the normalized Reynolds stress into the prestress.

The non-negative mapping between the prestress operator and the Reynolds stress depends on a scalar-valued turbulent transport time related to the relaxation of a Green's function associated with a convective–viscous parabolic differential operator and the relaxation of a two-point, space–time correlation related to turbulent velocity fluctuations. The preclosure equation also depends on a kinematic operator related to the average velocity gradient and a rotational operator related to the angular velocity of the frame.

The resulting universal realizable anisotropic prestress (URAPS-) closure is realizable for all non-rotating and rotating turbulent flows, provided the complementary transport equations for the turbulent kinetic energy and the turbulent dissipation are formulated to yield non-negative solutions. Experimental data and DNS results previously reported in the literature for non-rotating homogeneous simple shear and for non-rotating and rotating homogeneous decay are used to determine the closure constants. For rotating homogeneous simple shear, the URAPS-closure predicts the existence of self-similar states for finite positive and negative rotation numbers. The URAPS-closure for the NR-stress predicts anisotropic states consistent with expected behavior.

1. Introduction

1.1. Navier–Stokes equation in a rotating frame of reference

Turbulent flows of constant property Newtonian fluids in a rotating frame of reference are governed by the instantaneous Navier–Stokes (NS-) equation and the continuity equation (see Piquet, 1999, p. 18):

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\underline{F}} = -\nabla\left(\frac{p}{\rho} - \frac{\underline{x}\cdot(\underline{\underline{\Omega}}\cdot\underline{\underline{\Omega}}^T)\cdot\underline{x}}{2}\right) + \nu\nabla^2\underline{u} + \underline{g}, \tag{1.1}$$

$$\nabla\cdot\underline{u} = 0. \tag{1.2}$$

In Eq. (1.1), $\underline{\underline{F}}$ is a kinematic operator defined as $\underline{\underline{F}} \equiv \nabla\underline{u} + 2\underline{\underline{\Omega}}$, and $\underline{g}$ is the acceleration due to gravity. The anti-symmetric rotational operator $\underline{\underline{\Omega}}$ is related to the angular velocity $\underline{\Omega}$ of a rotating frame of reference (i.e., $\underline{\underline{\Omega}} = \underline{\underline{\underline{\varepsilon}}}\cdot\underline{\Omega}$). The symmetric and anti-symmetric components of the velocity gradient are defined as follows:

$$\nabla\underline{u} \equiv \frac{\nabla\underline{u} + (\nabla\underline{u})^T}{2} + \frac{\nabla\underline{u} - (\nabla\underline{u})^T}{2} = \underline{\underline{S}} + \underline{\underline{W}}. \tag{1.3}$$

Eq. (1.2) implies that the pressure distribution for constant density fluids satisfies a Poisson equation:

$$-\nabla^2\left(\frac{p}{\rho} - \frac{\underline{x}\cdot(\underline{\underline{\Omega}}\cdot\underline{\underline{\Omega}}^T)\cdot\underline{x}}{2}\right) = \nabla\cdot(\underline{u}\cdot\underline{\underline{F}}), \tag{1.4}$$

the independent variables $(\underline{x}, t)$ and the dependent variables $(\underline{u}, p)$ in Eqs. (1.1)–(1.4) are associated with a rotating frame of reference ($\underline{\underline{\Omega}} \neq \underline{\underline{0}}$). The same notation is used for the independent and dependent variables in a non-rotating frame ($\underline{\underline{\Omega}} = \underline{\underline{0}}$).

For large Reynolds numbers, Eqs. (1.1) and (1.2) subject to appropriate boundary conditions and initial conditions govern the behavior of turbulent flows of constant property Newtonian fluids.

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The instantaneous velocity and pressure fields are three-dimensional, spatially inhomogeneous, and temporally unsteady: $\underline{u} = \underline{u}(\underline{x}, t)$ and $p = p(\underline{x}, t)$. Small differences between two instantaneous velocity fields at $t = 0$ presumably cause large differences in the solutions to Eqs. (1.1) and (1.2) for $t > 0$. Instantaneous turbulent flows are neither temporally steady nor spatially uniform; however; average properties of the instantaneous fields are reproducible and may be statistically stationary and/or statistically homogeneous (Monin and Yaglom, 1971; Hinze, 1975; Pope, 2000).

1.2. Reynolds averaged Navier–Stokes equation in a rotating frame of reference

The reproducibility of low-order statistical properties of turbulent flows has partially motivated the use of statistical methods to study the behavior of constant property Newtonian fluids at large Reynolds numbers (see, esp., Piquet, 1999; Pope, 2000). If $\gamma \in \Gamma$ is used to designate a specific high Reynolds number solution to Eqs. (1.1) and (1.2), then an ensemble (or Reynolds) average of the velocity field and pressure field are formally represented by $\langle \underline{u} \rangle(\underline{x}, t) = \langle \underline{u}(\underline{x}, t; \gamma) \rangle$ and $\langle p \rangle(\underline{x}, t) = \langle p(x, t; \gamma) \rangle$, resp. For a fixed value of $\underline{x}$ and t, ensemble averages of Eqs. (1.1), (1.2), and (1.4) yield the following set of unclosed equations in a rotating frame of reference:

$$\frac{\partial \langle \underline{u} \rangle}{\partial t} + \langle \underline{u} \rangle \cdot \langle \underline{\underline{F}} \rangle = -\nabla \left(\frac{\langle p \rangle}{\rho} - \frac{\underline{x} \cdot (\underline{\underline{\Omega}} \cdot \underline{\underline{\Omega}}^T) \cdot \underline{x}}{2} \right) + \underline{g} + \nu \nabla^2 \langle \underline{u} \rangle - \nabla \cdot \langle \underline{u}' \underline{u}' \rangle, \tag{1.5}$$

$$\nabla \cdot \langle \underline{u} \rangle = 0, \tag{1.6}$$

$$-\nabla^2 \left(\frac{\langle p \rangle}{\rho} - \frac{\underline{x} \cdot (\underline{\underline{\Omega}} \cdot \underline{\underline{\Omega}}^T) \cdot \underline{x}}{2} \right) = +\nabla \cdot [\langle \underline{u} \rangle \cdot \langle \underline{\underline{F}} \rangle + \nabla \cdot \langle \underline{u}' \underline{u}' \rangle]. \tag{1.7}$$

In Eqs. (1.5) and (1.7), $\langle \underline{\underline{F}} \rangle \equiv \nabla \langle \underline{u} \rangle + 2\underline{\underline{\Omega}}$. Eq. (1.5), which is known as the Reynolds averaged Navier–Stokes (RANS-) equation, is exact, but unclosed. The RANS-equation governs the behavior of the first moment of the velocity distribution functional associated with an ensemble of velocity fields. The Reynolds average operation $\langle \cdot \rangle$ is linear and commutes with spatial and temporal derivatives within the frame of reference in which it is applied. The fluctuating velocity field is defined as $\underline{u}'(\underline{x}, t; \gamma) \equiv \underline{u}(\underline{x}, t; \gamma) - \langle \underline{u} \rangle(\underline{x}, t)$; and, the fluctuating pressure field is defined as $p'(\underline{x}, t; \gamma) \equiv p(\underline{x}, t; \gamma) - \langle p \rangle(\underline{x}, t)$. The Reynolds average of a mean field reproduces the mean field; therefore, $\langle \underline{u}'(\underline{x}, t; \gamma) \rangle = \underline{0}$ and $\langle p'(\underline{x}, t; \gamma) \rangle = 0$.

1.3. The normalized Reynolds stress

The single-point statistical correlation $+\langle \rho \underline{u}\, \underline{u}' \rangle (\equiv +\rho \langle \underline{u}' \underline{u}' \rangle)$ in the RANS-equation accounts for the mean flux of instantaneous momentum by velocity fluctuations relative to the mean velocity. This statistical correlation directly affects the mean velocity field as well as the mean pressure field. The normalized Reynolds momentum flux (or, equivalently, the normalized Reynolds (NR-) stress) is defined as

$$\underline{\underline{R}} \equiv \frac{\langle \underline{u}' \underline{u}' \rangle}{tr(\langle \underline{u}' \underline{u}' \rangle)}, \quad \underline{\underline{R}}^T = \underline{\underline{R}} \text{ and } tr(\underline{\underline{R}}) = 1. \tag{1.8}$$

The anisotropic component of the NR-stress $\underline{\underline{R}}$ is

$$\underline{\underline{b}} \equiv \underline{\underline{R}} - \tfrac{1}{3}\underline{\underline{I}}, \quad \underline{\underline{b}}^T = \underline{\underline{b}} \text{ and } tr(\underline{\underline{b}}) = 0. \tag{1.9}$$

By definition, the NR-stress is a non-negative operator inasmuch as

$$Q(\underline{z}) \equiv \underline{\underline{R}} : \underline{z}\underline{z} = \frac{\langle (\underline{z} \cdot \underline{u}')(\underline{u}' \cdot \underline{z}) \rangle}{tr \langle \underline{u}' \underline{u}' \rangle} \geq 0$$
$$\text{for all constant vectors } \underline{z} \in E^3 \ni \|\underline{z}\| = 1. \tag{1.10}$$

The above inequality is necessary and sufficient for the eigenvalues of $\underline{\underline{R}}$ to be non-negative and for the components of $\underline{\underline{R}}$ to satisfy Schwartz's inequalities (Schumann, 1977). A turbulent closure model for $\underline{\underline{R}}$ that satisfies Ineq. (1.10) for all turbulent flows in rotating and in non-rotating frames is *realizable.* As noted by Rung et al. (1999), Gatski and Jongen (2000), and Gatski (2004), many widely employed closure models, such as the "standard" k–ε model, do not satisfy Ineq. (1.10) for all turbulent flows and are, thereby, unsuitable closures for the RANS-equation.

In Eq. (1.10) implies that the two non-zero invariants of the normalized anisotropic stress $\underline{\underline{b}}$ (i.e., $\mathrm{II}_b \equiv tr(\underline{\underline{b}} \cdot \underline{\underline{b}})$ and $\mathrm{III}_b \equiv tr(\underline{\underline{b}} \cdot \underline{\underline{b}} \cdot \underline{\underline{b}})$) must fall on the bounded region defined by ABC in Fig. 1. This construction, which is referred to in the turbulence literature as the Lumley (L-) diagram (see Lumley, 1978; Pope, 2000, p. 394), holds for all anisotropic operators that are symmetric, normalized, and non-negative. A *realizable* turbulent closure model for $\underline{\underline{R}}$ has anisotropic invariants within or on the boundaries of the ABC-domain for all flows. Unfortunately, a turbulent model that is calibrated to be *realizable* for a specific flow may not be *realizable* for another flow. To avoid this dilemma, the condition that $\underline{\underline{R}}$ must be a non-negative operator should be explicitly incorporated into any precalibrated representation of $\underline{\underline{R}}$.

The shape of the quadratic form $Q(\underline{z})$ defined by Eq. (1.10) depends on the eigenvalues of $\underline{\underline{R}}$ and, thereby, the invariants of $\underline{\underline{b}}$ (i.e., II_b and III_b). Each realizable anisotropic state in Fig. 1 is associated with a quadratic form $Q(\mathrm{II}_b, \mathrm{III}_b)$. For example, oblate anisotropic states are on the AB-boundary, planar anisotropic states are on the BC-boundary, and prolate anisotropic states are on the AC-boundary. The realizable states defined by $\mathrm{III}_b = 0$ and $0 < \mathrm{II}_b < 2/9$ have the interesting feature that the eigenvalues of $\underline{\underline{R}}$ satisfy the following inequality: $0 \leq \lambda_{R1} \leq \lambda_{R2} = 1/3 \leq \lambda_{R3} \leq 2/3$. These states divide the L-diagram into two parts: the oblate-like states for which $\mathrm{III}_b < 0$; and, the prolate-like states for which $\mathrm{III}_b > 0$. Fig. 1 provides a useful means to compare experimental results, DNS results, and model predictions of anisotropic states associated with a non-negative, normalized, symmetric operator. The anisotropic states predicted by several commonly used algebraic Reynolds stress closure models for simple shear flows are noted below in order to underscore the need for the approach developed hereinafter.

1.4. "Eddy" viscosity closure

The "eddy" viscosity closure for $\underline{\underline{R}}$ was introduced by Boussinesq (1877) and continues to be widely used to study Eqs. (1.1) and (1.2). The underlying premise of the Boussinesq (B-) closure is that the mean field momentum flux caused by continuum scale turbulent fluctuations is analogous to the momentum flux caused by molecular scale fluctuations. Consequently, the B-closure assumes that (see Pope, 2000, p. 358)

$$\underline{\underline{R}}^B = \tfrac{1}{3}\underline{\underline{I}} + \underline{\underline{b}}^B, \quad \underline{\underline{b}}^B = -\tau_e \langle \underline{\underline{S}} \rangle, \quad \tau_e \equiv \frac{\nu_e}{k} = C_e \frac{k}{\varepsilon}, \tag{1.11}$$

where $k \equiv tr \langle \underline{u}' \underline{u}' \rangle / 2$ and $\varepsilon \equiv \nu \langle (\nabla \underline{u}') : (\nabla \underline{u}')^T \rangle$. According to Ineq. (1.10), the B-closure is *realizable* for all turbulent flows provided $3\tau_e \langle \underline{\underline{S}} \rangle : \underline{z}\underline{z} \leq 1$ for all constant vectors $\underline{z} \in E^3 \ni \|\underline{z}\| = 1$. Clearly, this condition implies that the "eddy" coefficient $C_e(\mathrm{II}_S, \mathrm{III}_S)$ depends on the invariants of the dimensionless mean strain rate operator, $k\langle \underline{\underline{S}} \rangle / \varepsilon$. The relationship between the mean strain rate and the normalized Reynolds stress expressed by Eq. (1.11) is presently used as a subgrid model for large eddy simulations (see Pope, 2000, p. 587) and as a closure model for the Reynolds stress in the RANS-equation. If C_e is assumed to be a *universal* constant (i.e., standard k–ε model), then the B-closure is clearly not *realizable* for all turbulent flows.

The B-closure implies that the "production" of turbulent kinetic energy, defined by $\mathscr{P}(\equiv -\langle \underline{u}' \underline{u}' \rangle : \langle \underline{\underline{S}} \rangle)$, is non-negative for all