Solid State Communications 218 (2015) 35–39

Contents lists available at ScienceDirect

Solid State Communications

journal homepage: www.elsevier.com/locate/ssc

A toy model for quantum spin Hall effect

S.A. Owerre [a,b,*], J. Nsofini [c,d]

[a] Département de Physique, Université de Montréal, Montréal, Québec, Canada H3C 3J7
[b] Perimeter Institute for Theoretical Physics, 31 Caroline St. N., Waterloo, Ontario, Canada N2L 2Y5
[c] Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
[d] Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

ARTICLE INFO

Article history:
Received 15 April 2015
Received in revised form
16 May 2015
Accepted 10 June 2015
Communicated by Xincheng Xie
Available online 17 June 2015

Keywords:
A. Topological insulators
D. Quantum spin Hall effect
D. Landau levels
D. Topological phase transitions

ABSTRACT

In this communication, we investigate a toy model of three-dimensional topological insulator surface, coupled homogeneously to a fictitious pseudospin-$\frac{1}{2}$ particle. We show that this toy model captures the interesting features of topological insulator surface states, which include topological quantum phase transition and quantum spin Hall effect. We further incorporate an out-of-plane magnetic field and obtain the Landau levels.

1. Introduction

In recent years, topological insulators (TIs) have captivated considerable attention of researchers [1–5,8]. These fascinating materials involve compounds such as Bi_2Se_3 or Bi_2Te_3, whose electronic bulk structure is an insulator with a finite gap separating the conduction band and the valence band, but their edges (for 2D TIs) or surfaces (for 3D TIs) have gapless states, which are protected by time-reversal symmetry (*TRS*). These states are robust to perturbations that do not break this symmetry. In recent years, these gapless states have been the focus of interest due to the simple Dirac-like Hamiltonian which describes them. However, breaking *TRS* introduces many interesting phenomena; this can be achieved by depositing a ferromagnet or a superconductor on the surface of a TI. Thus, the topologically protected surface state develops a gap, which leads to interesting electronic transport, which includes half quantized conductivity, Majorana bound states, etc. [1,11–15,17,18]. They also have potential technological applications in the field of spintronics [19,20].

In this communication, we study a toy model that captures some of the interesting features of surface states in topological insulators. Specifically, we study a fictitious pseudospin-$\frac{1}{2}$ particle interacting homogeneously with the surface of a TI. Assuming anisotropic "in-plane" and "out-of-plane" exchange interactions on the interface, say $\lambda_\parallel$ and $\lambda_\perp$, the coupled system exhibits a topological quantum phase transition as a function of $\xi = \lambda_\perp/\lambda_\parallel$. This phase transition is associated with a quantum phase transition point at $\xi = 1$, which separates two regions: $\xi < 1$, with two topologically protected Dirac points (semimetallic phase) and $\xi > 1$, which is fully gapped (quantized spin Hall phase). We further show that when the *z*-component of the pseudospin is conserved, the system decouples into two Hamiltonians which are related by *TRS*. The coupling term opens two gaps at the Γ point on the interface, which are degenerate but with opposite spin orientations. Each gap opening gives rise to a half quantized Hall conductivity. Upon applying an electric field, an opposite spin current is induced on each interface leading to a quantized spin Hall conductivity. Furthermore, we introduce a non-conserving term in the quantized spin Hall Hamiltonian by breaking the conservation of the *z*-component of the pseudospins. This leads to the obliteration of the quantized spin Hall conductivity. However, in this non-conserving regime we obtain a nontrivial topological spin Chern number by diagonalizing the projection of the pseudospin onto the occupied valence bands [21].

2. Simplest single Dirac point

Most of the interesting physics of 2D metals are captured by investigating the robustness of the band touching points (nodes). The bulk energy band near these points usually replicates a 2D Dirac

* Corresponding author.
E-mail addresses: solomon.akaraka.owerre@umontreal.ca (S.A. Owerre), jnsofini@uwaterloo.ca (J. Nsofini).

http://dx.doi.org/10.1016/j.ssc.2015.06.007
0038-1098/

Hamiltonian; thus these nodes are called Dirac points. The simplest form of Dirac point can be mapped out from this simple Hamiltonian:

$$H = v_F(\hat{z} \times \boldsymbol{\sigma}) \cdot \mathbf{k}, \tag{1}$$

where v_F is the Fermi velocity and $\sigma_i, i = x, y, z$, are the Pauli matrices representing the real spins of the surface states. The z-component is taken to be perpendicular to the plane of the 2D sample and $\mathbf{k} = (k_x, k_y)$ is the 2D surface Brillouin zone momentum. This Hamiltonian describes many known physical systems, such as the surface of a 3D topological insulator [8], and the Dirac point Hamiltonian in graphene in which the Pauli matrices are pseudospins [10]. Evidently, this Hamiltonian is invariant when the spin and momentum go to minus of their original values; in other words, the Hamiltonian possesses *TRS*, where the time reversal operator is given $\Theta = i\sigma_y\mathcal{K}$; $\mathcal{K}$ is a complex conjugation. The energy spectrum is trivial and given by $\mathcal{E}_s = s v_F\sqrt{k_x^2 + k_y^2}$; $s = \pm$. It is apparent that the two bands touch each other at $\mathbf{k} = 0$. The robustness of this Dirac point is guaranteed by any perturbation that preserves the *TRS* of Eq. (1). By breaking *TRS*, the degeneracy of the band might be lifted and many interesting phenomena can emerged. There are several terms that break *TRS*; for instance, $\Delta_x\sigma_x$ and $\Delta_y\sigma_y$ do break *TRS*; they correspond to Zeeman magnetic field contributions along the x and y directions respectively. However, addition of these terms to Eq. (1) simply shifts the position of the Dirac points to $\mathbf{k} = (0, -\Delta_x/v_F)$ and $\mathbf{k} = (\Delta_y/v_F, 0)$ respectively. Thus, the degeneracy is not lifted. The only *TRS* breaking term that lifts the degeneracy of the bands is the direct coupling of the surface electron spins to the Zeeman magnetic field term perpendicular to the plane, *i.e.*, $\Delta_z\sigma_z$. This term can also be generated by depositing a ferromagnet on the surface electrons [11,13,14]. It is evident that this contribution opens a gap of size $2|\Delta_z|$ at $\mathbf{k} = 0$. Consequently, when the Fermi energy lies between the gap, this leads to a half-quantized Hall conductivity [8,9]. In graphene, however, there are an even number of Dirac points. In this case, the charge Hall conductivity vanishes in the ordinary insulating state [5–7].

3. Toy model

As mentioned in the preceding sections, the surface of a 3D topological insulator possesses many interesting features when *TRS* is explicitly broken. In this section, we will consider a 3D topological insulator whose surface is homogeneously coupled to a fictitious pseudospin-$\frac{1}{2}$ particle; the low-energy effective Hamiltonian can be written as

$$\mathcal{H} = v_F(\hat{z} \times \boldsymbol{\sigma}) \cdot \mathbf{k} - \lambda_\parallel \tau_x\sigma_x - \lambda_\perp \tau_z\sigma_z, \tag{2}$$

where σ_i and τ_i $(i = x, y, z)$ are the Pauli matrices acting on the topological insulator space and the fictitious pseudospin-$\frac{1}{2}$ particle space respectively. The coupling constants $\lambda_\parallel$ and $\lambda_\perp$ are the anisotropic in-plane and out-of-plane homogeneous exchange interactions. In most cases of physical interest, such interaction is usually inhomogeneous, *i.e.*, the last two terms in Eq. (2) should contain a delta function. Thus, Eq. (2) does not describe any known physical system. However, it is possible that it might be applicable to a pseudo-quantum qubit, but we are not interested in any specific system, instead we will consider it as a toy model that captures some of the interesting physics of TI surface states. Although our model does not describe any known physical system, it possesses some features that are similar to other models that describe known systems. This is the main purpose of this communication. We will consider Eq. (2) as the basis of our investigation in this Communication. It is apparent that Eq. (2) explicitly breaks *TRS*, *i.e*, $[\Theta, \mathcal{H}] \neq 0$, where $\Theta = \tau_x \otimes i\sigma_y\mathcal{K}$. Due to the fictitious pseudospin-$\frac{1}{2}$ particle, Eq. (2) is obviously a 4-band model.

4. Topological quantum phase transition

Most known systems such as graphene and thin film topological insulators possess topological quantum phase transition. This is manifested at the gap closing point which separates an ordinary insulator from a quantized Hall insulator [16,26]. These phases are usually distinguished by a topological invariant quantity called the Chern number. It is generally defined as [25]

$$\mathcal{C} = \frac{1}{24\pi^2}\epsilon_{\alpha\beta\gamma}\,\mathrm{Tr}\int d^3k\,\mathcal{G}\partial_{k_\alpha}\mathcal{G}^{-1}\mathcal{G}\partial_{k_\beta}\mathcal{G}^{-1}\mathcal{G}\partial_{k_\gamma}\mathcal{G}^{-1}, \tag{3}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric tensor, $\alpha = 0, x, y, z$, etc., labels the components of a four-vector, and Matsubara Green's function can be written as

$$\mathcal{G}^{-1}(i\omega_n, \mathbf{k}) = i\omega_n\mathbf{I}_{4\times4} - \mathcal{H}. \tag{4}$$

The identity $\mathbf{I}_{4\times4}$ is a 4×4 matrix and $k_\alpha = (i\omega_n, \mathbf{k})$ is the momentum four-vector. In principle the Chern number can be computed for any model of interest. One finds that it has a unique value in each phase, which is immutable by any smooth deformation of the system (provided the gap does not close). In general, the emergence of quantum phase transition in electron systems requires the violation of CPT symmetry [25], *i.e.*, charge conjugation, parity, and time-reversal symmetries. It is evident that our model in Eq. (2) explicitly breaks parity and time-reversal symmetries. Thus it can capture quantum phase transitions similar to those predicted in known systems. In order to see this, we diagonalize Eq. (2) and find that the eigenvalues are given by

$$\mathcal{E}_{s\eta} = (-1)^\eta\sqrt{v_F^2k_x^2 + \left(\sqrt{v_F^2k_y^2 + \lambda_\perp^2} + s\lambda_\parallel\right)^2}, \tag{5}$$

where $s = \pm$ and $\eta = 0, 1$. There are two energy sectors $\mathcal{E}_{+\eta}$ and $\mathcal{E}_{-\eta}$. In each sector there are two bands with $\eta = 0$ being the conduction band and $\eta = 1$ being the valence band.

The $\mathcal{E}_{-\eta}$ bands have two Dirac points ($\mathcal{E}_{+\eta}$ bands are always gapped) at $\pm\mathbf{K}$, where

$$\mathbf{K} = \left(0, \frac{\lambda_\parallel}{v_F}\sqrt{1-\xi^2}\right), \quad \xi = \lambda_\perp/\lambda_\parallel. \tag{6}$$

The topological quantum phase transition as a function of ξ can be understood as follows. At the point $\xi = 0$, the system is a semimetal with two Dirac points separated by the wave vector $v_Fk_y = 2\lambda_\parallel$ along the y-direction. The semimetallic phase is controlled by the in-plane exchange interaction. Provided $\xi < 1$, these two Dirac points remain intact, separated in momentum space by a wave vector $v_Fk_y = 2\lambda_\parallel\sqrt{1-\xi^2}$. They are topologically protected and cannot be eliminated by changing the parameters of the

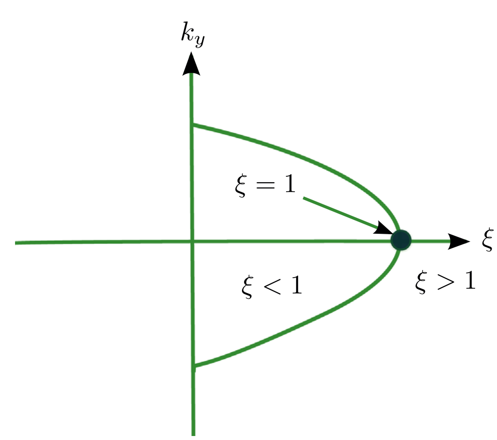

Fig. 1. Color online. The phase transition in (ξ, k_y) space with $v_F = 1 = \lambda_\parallel$. The quantum phase transition point $\xi = 1$ separates two regions: $\xi < 1$, with two Dirac points and $\xi > 1$, which is fully gapped; see Fig. 2.