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Hydrodynamic force between two hard spheres tangentially translating in a power-law fluid

Wenbin Huang[a], Hongyan Li[b], Yong Xu[a], Guoping Lian[c,*]

[a]*Department of Applied Mechanics, China Agricultural University, Beijing 100083, PR China*
[b]*Department of Transport Engineering, Shandong University of Technology, Zibo 255049, PR China*
[c]*Unilever R&D Colworth, Sharnbrook, Bedford MK44 1LQ, UK*

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Abstract

The hydrodynamic interaction between two hard spheres tangentially translating in a power-law fluid is investigated. By considering the gap between the two spheres being sufficiently small such that the Reynolds' lubrication theory applies, an analytical equation to the pressure in the gap is obtained using truncated Fourier series. To a good approximation, the pressure equation can be further simplified. The simplified approximate equation over-predicts the pressure for shear thickening fluid ($n > 1$) but under-predicts the pressure for shear-thinning fluid ($n < 1$). However, the errors in the predicted tangential force and moment are relatively small. In particular, for a Newtonian fluid, the accurate solution and the simplified approximate solution degenerate to the asymptotic solution of Goldman et al. [1967. Slow viscous motion of a sphere parallel to a plane wall-motion through a quiescent fluid. Chemical Engineering Science 22, 637–651.] and O'Neill and Stewartson [1967. On the slow motion of a sphere parallel to a nearby plane wall. Journal of Fluid Mechanics 27, 705–724.]. Both solutions predict that for shear thickening fluid ($n > 1$), the hydrodynamic force converged in the inner region of the gap between the two spheres and the contribution from the outer region is sufficiently small. For shear thinning fluid ($n < 1$), the contribution from the outer region is also significant.

Keywords: Hydrodynamic interaction; Thin-film lubrication; Squeeze flow; Inter-particle viscous force

1. Introduction

For concentrated dispersions and suspensions, the hydrodynamic interaction between non-colloidal particles plays an important role in their rheological behaviour, such as shear induced particle migration and diffusion (Leighton and Acrivos, 1987; Cunca and Hinch, 1996). Some experimental studies (e.g. Ohl and Gleissle, 1993) on concentrated suspensions have shown that, when the inter-particle forces are dominated by hydrodynamic effects, the flow behaviour can be described by a shift factor for the reduced shear rate under constant shear stress, with the shift factor being a function of the concentration and characteristics of the suspended particles. These studies gave clear evidence that the rheological behaviour scales with the hydrodynamic force between suspended particles. Formulation of the constitutive relationship of the reduced shear rate shift factor has been mostly relied on empirical equations (e.g. Kamal and Mutel, 1985). A limiting factor for the development of generalised rheological equations for concentrated suspensions is due to the lack of analytical solutions to the pair-wise hydrodynamic forces between solid particles in non-Newtonian fluids.

Recent theoretical studies on the rheological behaviour of multi-phase materials use computational simulations including Stokesian Dynamics (Brady, 2001; Foss and Brady, 2000, Schaink et al., 2000), Discrete Element Method (Lian et al., 1998) and Lattice-Boltzmann (Ladd and Verberg, 2001; Lee and Ladd, 2002). With the Stokesian Dynamics and Discrete Element simulations, it is necessary to implement more physically realistic inter-particle hydrodynamic forces. Most computer simulations on the rheological properties of colloidal and concentrated suspensions used relatively simple hard sphere interaction model (e.g. Bergenholtz et al., 2002).

* Corresponding author. Tel.: +44 1234 222741; fax: +44 1234 222259.
E-mail address: guoping.lian@unilever.com (G. Lian).

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This clearly limits wider applications of the computational method.

The hydrodynamic interaction between two near-touching spherical particles dispersed in a fluid can be resolved into three components of normal, tangential and spinning. The normal component of the relative approach between two near-touching spheres along their common axis is often referred to as squeeze flow. The problem has been studied theoretically by many researchers (Adams and Edmondson, 1987; Rodin, 1996, Xu et al., 2001). A closed-form solution of the viscous force arising from the squeeze flow of a power-law fluid between two rigid spheres was recently derived by Lian et al. (2001). There have been relatively fewer studies on the hydrodynamic interaction between two tangentially translating hard spheres. The problem is asymmetrical. O'Neill and Stewartson (1967) studied the problem of a sphere translating parallel to a plane wall bounded by a Newtonian fluid, and obtained a numerical solution. Goldman et al. (1967) studied the same problem and presented an asymptotic solution. However, their solution to the pressure distribution exhibits singularity at the centreline of the inner gap. This paper presents an analytical study on the hydrodynamic interaction between two tangentially translating spheres in a power-law fluid. The gap between the two spheres is considered to be sufficiently small such that the lubrication theory applies.

The analytical solution is derived with the pressure distribution in the gap represented by truncated Fourier series. The Perturbation method similar to Goldman et al. (1967) is used. Integration of the pressure allows the closed-form solution to the corresponding tangential traction and moment obtained.

Tangential interaction between two near touching spheres in a complex fluid is also of relevance to a number of industrial and process engineering problems including thin-film lubrication (Luo et al., 1996), single and double particle microrheology (Levine and Lubensky, 2000), sedimentation (Gheissary and van den Brule, 1996; Schmeeckle et al., 2001) and structure formation in suspensions (Scirocco et al., 2004). In those applications, the hydrodynamic interaction between suspended particles plays an important role in determining the complex dynamics including particle chaining, inter-particle collision and aggregation.

2. Governing equations

Consider a rigid sphere of radius R_1 translating parallel to another rigid sphere of radius R_2 with a relative velocity, U, in x-direction, shown in Fig. 1(a). The gap between the two spheres is sufficiently small and the lubrication theory is considered to apply. The problem can be extended to the problem of a sphere translating parallel to a wall when the radius of the one sphere is set to infinity. For convenience, a cylindrical coordinate system (r, φ, z) is chosen. The minimum gap at the centreline between the two spheres is s_0 and the fluid in the gap is considered as a power-law fluid. The general form of the constitutive relationship for a power-law fluid is given as follows:

$$\sigma_{ij} = \mu\dot{\gamma}_{ij} = K\dot{\gamma}^{n-1}\dot{\gamma}_{ij} \quad \text{with } \dot{\gamma} = \sqrt{\tfrac{1}{2}\dot{\gamma}_{ij}\dot{\gamma}_{ij}}, \tag{1}$$

where σ_{ij} is the shear stress tensor ($i, j=1, 2$ for 2D and 1, 2, 3 for 3D), $\mu = K\dot{\gamma}^{n-1}$ is the apparent viscosity, K is the flow consistency, n is the flow index, and $\dot{\gamma}$ is related to the second invariant of strain rate tensor $\dot{\gamma}_{ij} = 2\dot{\varepsilon}_{ij}$. The above constitutive relationship of power-law applies to both shear thinning ($n < 1$) and shear thickening ($n > 1$) fluid. In particular, for $n = 1$, we have the Newtonian fluid.

We consider the gap between the two spheres be sufficiently small such that the lubrication assumption applies. The momentum equation of the interstitial fluid between the two spheres is reduced to the following form:

$$\frac{\partial p}{\partial r} = \frac{\partial \sigma_{rz}}{\partial z},$$

$$\frac{1}{r}\frac{\partial p}{\partial \varphi} = \frac{\partial \sigma_{\varphi z}}{\partial z},$$

$$\frac{\partial p}{\partial z} = 0, \tag{2}$$

where σ_{rz} and $\sigma_{\varphi z}$ are the stress components, respectively and p is the fluid pressure with $p = p(r, \varphi)$.

The continuity equation of the fluid in the gap satisfies the following relationship:

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} = 0, \tag{3}$$

where u_r, u_φ and u_z are the velocity components.

Generally, for flow in 2D and 3D flow problems, standard tensor notation is used. The constitutive relationship of Eq. (1) is generally expressed in terms of the second invariant of the strain rate tensor. For the tangential translation between two spheres, the leading terms of the shear strain rate are $\dot{\gamma}_{rz} = \partial u_r/\partial z$ and $\dot{\gamma}_{\varphi z} = \partial u_\varphi/\partial z$. The second invariant of the strain rate tensor, I_2, can be approximated as

$$I_2 \approx \frac{1}{4}(\dot{\gamma}_{rz}^2 + \dot{\gamma}_{\varphi z}^2) \approx \frac{1}{4}\dot{\gamma}^2 = \frac{1}{4}\left[\left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_\varphi}{\partial z}\right)^2\right]. \tag{4}$$

Since the gap is considered to be sufficiently small, i.e., $s_0 \to$ O(1), the two near surfaces S_1 and S_2 can be approximated by

$$S_1: \quad z = z_1(r) = s_0 + \frac{r^2}{2R_1},$$

$$S_2: \quad z = z_2(r) = -\frac{r^2}{2R_2}. \tag{5}$$

It follows that the separation distance between the two near surfaces of the two spheres can be derived as follows:

$$s(r) = z_1(r) - z_2(r) = s_0 + \frac{r^2}{2R^*}, \tag{6}$$