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Circular Couette flow of temperature-dependent materials: asymptotic solutions in the presence of viscous heating

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INTRODUCTION

The interaction between viscous heating and flow is of importance in a number of applications involving flow of viscous fluids with temperature-dependent properties. These include polymer processing, tribology and lubrication, food processing, on-line measurements, instrumentation and viscometry. In the latter area, viscous heating is always a possible, and frequently significant, source of error in viscometric measurements at high shear rates, particularly with rotational viscometers, where the entire sample is sheared continuously for the duration of the measurement. When the thermal conductivity of the fluid is constant and its viscosity an exponential function of temperature, analytical solutions for planar Couette flow (including the effect of viscous heating) have been presented by Turian and Bird (1963) and for circular Couette flow by Gavis and Lawrence (1968). Bird *et al.* (1960), Turian and Bird (1963) and Turian (1965) have presented a methodology for obtaining approximate analytical solutions to the problem of combined flow and heat transfer in planar Couette flow when both, the viscosity and the thermal conductivity, are polynomial functions of temperature. The present contribution applies the same method and develops series solutions, up to second order in the Brinkman number, for circular Couette flow. This flow has received attention recently in the study of microstructure evolution during processing of concentrated suspensions (Chow *et al.*, 1993) or in the study of fibre motion in non-homogeneous flow fields. Many of the test fluids used in such studies are Newtonian with relatively high, temperature-dependent viscosity, and there is a need for explicit solutions for the velocity field which will take into account viscous heating and the effect of temperature on the transport properties of the fluid. Such solutions can be linked, as de-coupled explicit modules, to numerical, microstructure-oriented models, such as the ones presented by Phillips *et al.* (1992) for particulate suspensions, or by Ranganathan and Advani (1993) for the evolution of the orientation in fibre suspensions.

THEORY

We consider steady, incompressible flow in a circular Couette device consisting of two concentric cylinders of radii R and κR ($\kappa < 1$), of which the inner one is stationary and the outer is rotating with constant angular velocity Ω. The equations of motion and energy reduce to:

$$\frac{1}{x^2}\left\{\frac{\partial}{\partial x}\left\{\frac{\mu}{\mu_0}x^3\frac{\partial}{\partial x}\left(\frac{u}{x}\right)\right\}\right\} = 0 \tag{1}$$

$$\frac{1}{x}\frac{\partial}{\partial x}\left\{\frac{k}{k_0}x\frac{\partial\Theta}{\partial x}\right\} + Br\frac{\mu}{\mu_0}\left[x\frac{\partial}{\partial x}\left(\frac{u}{x}\right)\right]^2 = 0 \tag{2}$$

where the following non-dimensionalisation has been applied:

$$\Theta = \frac{T - T_0}{T_0}, \qquad x = \frac{r}{R}, \qquad u = \frac{u_\theta}{\Omega R}, \qquad Br = \frac{\mu_0(\Omega R)^2}{k_0 T_0}. \tag{3}$$

In eq. (3) (Br) is the Brinkman number, which is a measure of the heat generated by viscous heating as compared to the heat conducted through the material. The subscript (0) indicates property values corresponding to the reference temperature T_0. The Brinkman number is not sufficient in quantifying the effect of viscous heating on the velocity profile in the presence of a temperature-dependent viscosity (Pearson, 1985). In this case, the Nahme number [$Na = Br(\partial\mu/\partial T)(T_0/\mu_0)$] is the appropriate scale. This is introduced in later sections where viscous heating is quantified by the use of the product $Br\,\beta$ (e.g. Tables 1 and 2 below) instead of (Br) alone, where (β) is the leading term in the viscosity model [eq. (5) below].

Following the methodology proposed by Bird *et al.* (1960), Turian and Bird (1963) and Turian (1965), we seek approximate solutions for $u(x)$ and $\Theta(x)$ of the form:

$$\frac{u(x)}{x} = u_0(x) + \sum_{n=1}^{N} u_n(x)\,Br^n,$$

$$\Theta(x) = \Theta_0(x) + \sum_{n=1}^{N} \Theta_n(x)\,Br^n \tag{4}$$

for fluids whose transport properties, namely the viscosity and the thermal conductivity, are arbitrary polynomials of temperature:

$$\frac{k}{k_0} = 1 + \sum_{i=1}^{I}\alpha_i\Theta^i, \qquad \frac{\mu_0}{\mu} = 1 + \sum_{i=1}^{I}\beta_i\Theta^i \tag{5}$$

where the order of the approximations are not necessarily equal and where, in practice, the coefficients α_i and β_i will be determined by fitting experimental data. This presentation of material property data is very common in the process industries and has been a motivation for the development of the solutions presented in this study. In the absence of viscous heating ($Br = 0$) the system is isothermal and therefore $\Theta_0(x)$ in eq. (4) is identically zero. We consider the following boundary conditions:

$$\text{At } x = \kappa \text{ (inner cylinder surface):} \quad u = 0 \text{ and } \Theta = 0 \tag{6a}$$

$$\text{At } x = 1 \text{ (outer cylinder surface):} \quad u = 1 \text{ and } \Theta = 0. \tag{6b}$$

It should be noted that the solution procedure adopted in this work is quite general and can admit derivative as well as non-zero Dirichlet conditions on either of the cylinder surfaces. The problem of an adiabatic inner cylinder has been considered by Papathanasiou *et al.* (1997).

RESULTS

The details of the solution procedure are similar to those of the planar problem (Louwagie, 1994) and are omitted for the sake of brevity. The second-order solution for the circular Couette flow follows:

Velocity:

$$u_0(x) := \frac{(\kappa^2 - x^2)}{(\kappa^2 - 1)x^2} \tag{7}$$

$$u_1(x) = \frac{\kappa^4 \beta_1}{2(\kappa^2 - 1)^3}\left[\frac{\kappa^2 - 1}{x^2} + 1 + \frac{\ln(x)}{x^2}\frac{2(1-\kappa^2)}{\ln(\kappa)} - \frac{\kappa^2}{x^4}\right] \tag{8}$$

Table 1. The norm of the difference between numerical and series solutions for flow in a circular Couette for a range of values of the product $Br\,\beta$. $\kappa = 0.5$, $\beta_1 = \beta_2 = \beta = 2$, $\alpha_1 = 0.5$

$Br\,\beta$	norm (u_R)	norm (Θ_R)
2.0	0.0071	0.6573
4.0	0.0565	2.560
5.0	0.1095	3.948
6.0	0.1875	5.6133
8.0	0.4353	9.7331

$$u_2(x) := \frac{1}{24(\kappa^2 - 1)^5 x^6 \ln(\kappa)^2}(M_{10}(x)\kappa^{10} + M_8(x)\kappa^8 + M_6(x)\kappa^6 + M_4(x)\kappa^4 + M_2(x)\kappa^2 + M_0(x)) \tag{9}$$

where the coefficient functions $M_{10}(x)$–$M_0(x)$ are given by

$$M_{10}(x) := -12x^4(\beta_1\alpha_1 - 2\beta_2)\ln(x)^2 + 12x^2[A^{(10)}x^2 + B^{(10)}]\ln(x) + C^{(10)}x^4 + D^{(10)}x^2 - E^{(10)} \tag{10}$$

with

$$\begin{aligned} A^{(10)} &:= (-1 + 2\ln(\kappa))(\beta_1\alpha_1 - 2\beta_2) \\ B^{(10)} &:= 2\beta_2 \ln(\kappa) + \beta_1^2 \ln(\kappa) - \beta_1\alpha_1 \ln(\kappa) \\ C^{(10)} &:= 12\beta_2 + 12\beta_1\alpha_1 \ln(\kappa) - 6\beta_1\alpha_1 + 24\beta_2 \ln(\kappa)^2 - 24\beta_2 \ln(\kappa) \\ D^{(10)} &:= 3\ln(\kappa)(-8\beta_2 \ln(\kappa) + 4\beta_1\alpha_1 \ln(\kappa) - 4\beta_1^2 \ln(\kappa) + 2\beta_2 + 5\beta_1^2 - \beta_1\alpha_1) \\ E^{(10)} &:= 2\ln(\kappa)^2(2\beta_1\alpha_1 - 4\beta_2 - \beta_1^2) \end{aligned} \tag{11}$$

$$M_8(x) := 24x^4(-2\beta_2 + \beta_1\alpha_1)\ln(x)^2 + 3x^2[A^{(8)}x^2 + B^{(8)}]\ln(x) + 3x^2[C^{(8)}x^2 + D^{(8)}] \tag{12}$$

with

$$\begin{aligned} A^{(8)} &= -8\beta_1\alpha_1 \ln(\kappa) + 16\beta_2 \ln(\kappa) - 16\beta_2 + 8\beta_1\alpha_1 \\ B^{(8)} &= -4\beta_1^2 \ln(\kappa) - 8\beta_2 \ln(\kappa) + 4\beta_1\alpha_1 \ln(\kappa) \\ C^{(8)} &= 4\beta_1\alpha_1 - 4\beta_1\alpha_1 \ln(\kappa) + 8\beta_2 \ln(\kappa) - 8\beta_2 \\ D^{(8)} &= \beta_1\alpha_1 \ln(\kappa) - 2\beta_2 \ln(\kappa) - 5\beta_1^2 \ln(\kappa) \end{aligned} \tag{13}$$

$$M_6(x) := -6x^4(2\ln(x) + 1 + 2\ln(x)^2)(-2\beta_2 + \beta_1\alpha_1) \tag{14}$$

$$M_4(x) := 6\ln(\kappa)x^2\beta_1\left[A(x^2 - 2\ln(\kappa)x^2 + 3\ln(\kappa) + 2x^2\ln(x)) + \frac{\beta_1\kappa^4}{2}(\kappa^2 + 1)x^2(1 + 2\ln(x))\right] \tag{15}$$

$$M_2(x) := 6\beta_1 x^4 \ln(\kappa)\left\{4\ln(\kappa)E - (1 + 2\ln(x)) \times \left[\frac{\beta_1\kappa^4}{2}(\kappa^2 + 1) + A\right]\right\} \tag{16}$$

$$M_0(x) := 12\ln(\kappa)^2 x^4(-B + 2x^2 D). \tag{17}$$

Table 2. Comparison between numerical (u_{an}, Θ_{an}) and series (u_s, Θ_s) solutions for temperature and velocity across the gap of a wide-gap Couette for two values of the product ($Br\,\beta$). $\kappa = 0.5$, $\beta_1 = \beta_2 = \beta = 2$, $\alpha_1 = 0.5$

	$Br\,\beta = 3$				$Br\,\beta = 5$			
x	u_s	u_{an}	Θ_s	Θ_{an}	u_s	u_{an}	Θ_s	Θ_{an}
0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.55	0.1202	0.1201	0.0418	0.0424	0.1166	0.1162	0.0639	0.0667
0.6	0.2374	0.2374	0.0645	0.0655	0.2337	0.2334	0.0989	0.1030
0.65	0.3498	0.3498	0.0742	0.0753	0.3476	0.3475	0.1141	0.1186
0.7	0.4566	0.4566	0.0751	0.0762	0.4563	0.4563	0.1154	0.1200
0.75	0.5580	0.5581	0.0698	0.0708	0.5594	0.5595	0.1073	0.1115
0.8	0.6544	0.6544	0.0602	0.0611	0.6567	0.6569	0.0925	0.0962
0.85	0.7462	0.7463	0.0476	0.0483	0.7488	0.7491	0.0732	0.0762
0.9	0.8341	0.8341	0.0330	0.0335	0.8363	0.8366	0.0507	0.0528
0.95	0.9185	0.9186	0.0170	0.0172	0.9198	0.9201	0.0261	0.0271
1	1.0	1.0	0.0	0.0	1.0	1.0	0.0	0.0