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Higher-order asymptotic solution for the fatigue crack growth problem based on continuum damage mechanics

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Abstract

In the paper an analytical solution of the nonlinear eigenvalue problem arising from the fatigue crack growth problem in a damaged medium in coupled formulation is obtained. In order to evaluate the mechanical behavior in the vicinity of a growing fatigue crack for plane strain and plane stress conditions of mode I the asymptotic governing equations are derived and analyzed by the light of Continuum Damage Mechanics. It is shown that the growing fatigue crack problem can be reduced to the nonlinear eigenvalue problem. The perturbation technique for solving the nonlinear eigenvalue problem is used. The method allows to find the analytical formula expressing the eigenvalue as the function of parameters of the damage evolution law. The eigenvalues of the nonlinear eigenvalue problem are fully determined by the exponents of the damage evolution law. The higher-order asymptotic expansions of the angular functions determining the stress and continuity fields in the neighborhood of the crack tip are given. The asymptotic expansions of the angular functions permit to find the closed-form solution for the problem. The higher order stress, strain and continuity asymptotic fields in the vicinity of the fatigue growing crack are either obtained, in which analytical expressions of the higher order exponents and angular distribution functions (eigenfunctions) of the near tip stress and continuity fields are derived.

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1. Introduction

The asymptotic analysis of the near crack tip stress field is an integral part of fracture mechanics analysis for both linear elastic media and nonlinear media. In nonlinear fracture mechanics the eigenfunction expansion method results in nonlinear eigenvalue problems (Murakami (2012); Stepanova (2008)). The most commonly encountered approaches to tackle the nonlinear eigenvalue problem are the Runge-Kutta method and FEM approach. However, the numerical integration of the equations realized by the Runge-Kutta method in conjunction with the shooting method generally becomes multiparametric. Thus, the numeric results still require further verification. That is the reason why much attention is currently given to analytical methods of solution of nonlinear fracture mechanics problems as

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a whole and, specifically, to methods of solution of nonlinear eigenvalue problems (Stepanova (2008, 2009)). One of promising approaches is the perturbation theory technique applied to a wide variety of static and dynamic solid mechanics problems. In this contribution an analytical solution of the nonlinear eigenvalue problem arising from the fatigue crack growth problem in a damaged medium in coupled formulation is obtained. The original statement of the problem considered here is proposed by Zhao and Zhang (1995) where it is noted that further theoretical efforts are wanted in the respects of appropriate determination of higher ordered asymptotic studies. The present study continues the analysis of Zhao and Zhang (1995); Stepanova and Igonin (2013) and gives an attempt to construct the higher-order asymptotic expansions of stresses, strains and continuity in the vicinity of the growing fatigue crack and to derive the analytical expressions of the higher-order crack-tip fields.

2. Basic formulation

Consider mode I deformations around a fatigue growing crack in a damaged material. We will neglect the anisotropic effects induced by damage. Under assumption of linear elasticity the constitutive equation of damaged materials is $\varepsilon_{ij} = (1+\nu)\sigma_{ij}/(E\psi) - \nu\sigma_{kk}\delta_{ij}/(E\psi)$, where ε_{ij} are infinitesimal strains, σ_{ij} are stresses, E is Young's modulus, ν is Poisson's ratio of undamaged material, ψ is a scalar internal variable that characterizes damage ($0 \leqslant \psi \leqslant 1$), where $\psi = 1$ and $\psi = 0$ signify the initial undamaged state and the final completely damaged state, respectively, Murakami (2012); Kuna (2013). Following Zhao and Zhang (1995) we assume that the cumulative damage evolution equation has the form

$$d\psi/dN = \left\{ -c\,(\sigma_e/\psi)^m\,\psi^{m-n}, \text{ for } \sigma_e \geqslant \sigma_{th}\psi; \quad 0 \text{ for } \sigma_e < \sigma_{th}\psi \right. , \tag{1}$$

where N is the number of cycles, c, m, n, σ_{th} are positive material parameters. The constitutive equations for plane stress conditions are given as

$$\varepsilon_{rr} = (\sigma_{rr} - \nu\sigma_{\theta\theta})/(E\psi), \qquad \varepsilon_{\theta\theta} = (\sigma_{\theta\theta} - \nu\sigma_{rr})/(E\psi), \qquad \varepsilon_{r\theta} = (1+\nu)\sigma_{r\theta}/(E\psi). \tag{2}$$

The equilibrium equations $r\sigma_{rr,r} + \sigma_{r\theta,\theta} + (\sigma_{rr} - \sigma_{\theta\theta}) = 0$, $r\sigma_{r\theta,r} + \sigma_{\theta\theta,\theta} + 2\sigma_{r\theta} = 0$ can be satisfied by introducing the Airy stress function $\chi(r,\theta) : \sigma_{\theta\theta}(r,\theta) = \chi_{,rr}$, $\sigma_{rr}(r,\theta) = \chi_{,r}/r + \chi_{,\theta\theta}/r^2$, $\sigma_{r\theta}(r,\theta) = -\left(\chi_{,\theta}/r\right)_{,r}$. The compatibility equation is

$$2\left(r\varepsilon_{r\theta,\theta}\right)_{,r} = \varepsilon_{rr,\theta\theta} - r\varepsilon_{rr,r} + r(r\varepsilon_{\theta\theta})_{,rr}. \tag{3}$$

The solution of Eqs. (1) – (3) should satisfy the traditional traction free boundary conditions on crack surfaces $\sigma_{\theta\theta}(r, \theta = \pm\pi) = 0$, $\sigma_{r\theta}(r, \theta = \pm\pi) = 0$.

3. Asymptotic solution. Eigenvalues and eigenfunctions

The Airy stress function $\chi(r,\theta)$ and the continuity parameter $\psi(r,\theta)$ in the damage accumulation process zone (as $r \to 0$) are assumed in the separated form as follows

$$\chi(r,\theta) = \sum_{k=0}^{\infty} \alpha_k r^{\lambda_k+2} f_k(\theta), \qquad \psi(r,\theta) = \sum_{k=0}^{\infty} \beta_k r^{\mu_k} g_k(\theta), \tag{4}$$

where $f_k(\theta)$, $g_k(\theta)$, $k = 0, 1, ...$ are the eigenfunctions that define the angular dependence; α_k, β_k are constants depending on the geometry and loading conditions, λ_k, μ_k are the eigenvalues that govern the behavior of stresses and continuity at the tip of a crack. The asymptotic expansions of the stress components take the form $\sigma_{ij}(r,\theta) = \sum\limits_{k=0}^{\infty} \alpha_k r^{\lambda_k} \sigma_{ij}^{(k)}(\theta)$, where $\sigma_{\theta\theta}^{(k)}(\theta) = (\lambda_k + 2)(\lambda_k + 1) f_k(\theta)$, $\sigma_{rr}^{(k)}(\theta) = (\lambda_k + 2) f_k(\theta) + f_k''(\theta)$, $\sigma_{r\theta}^{(k)}(\theta) = -(\lambda_k + 1) f_k'(\theta)$, $k = 0, 1, ...$ The constitutive equations (2) and the asymptotic expansions introduced allow us to develop the asymptotic expansions for the strain components near the crack tip as $r \to 0$

$$\varepsilon_{rr} = \frac{1}{E} \sum_{j=0}^{\infty} \frac{\alpha_j}{\beta_0} r^{\lambda_j - \mu_0} \varepsilon_{rr}^{(j)}(\theta), \quad \varepsilon_{\theta\theta} = \frac{1}{E} \sum_{j=0}^{\infty} \frac{\alpha_j}{\beta_0} r^{\lambda_j - \mu_0} \varepsilon_{\theta\theta}^{(j)}(\theta), \quad \varepsilon_{r\theta} = \frac{1+\nu}{E} \sum_{j=0}^{\infty} \frac{\alpha_j}{\beta_0} r^{\lambda_j - \mu_0} \varepsilon_{r\theta}^{(j)}(\theta), \tag{5}$$