Vacuum 84 (2010) 734–737

Contents lists available at ScienceDirect

Vacuum

journal homepage: www.elsevier.com/locate/vacuum

Numerical analysis of oscillatory Couette flow of a rarefied gas on the basis of the linearized Boltzmann equation

Toshiyuki Doi

Department of Applied Mathematics and Physics, Tottori University, Tottori, 680-8552 Japan

ARTICLE INFO

Article history:
Received 27 October 2008
Received in revised form
24 April 2009
Accepted 8 October 2009

PACS:
47.45.-n
47.45.Ab
07.10.Cm

Keywords:
Rarefied gas
Boltzmann equation
Forced oscillation

ABSTRACT

The unsteady motion of a rarefied gas between two parallel plates, one of which is oscillating in its plane, is studied based on the linearized Boltzmann equation for a hard sphere molecular gas. The detailed numerical solution for a wide range of gas rarefaction and oscillatory frequency is presented. The transition of the solution from low to high frequencies under various degrees of gas rarefaction is discussed.

1. Introduction

The flow of a rarefied gas has been investigated extensively for more than five decades, and the behavior of various fundamental flows has been clarified [1]. The results of these studies are successfully applied to vacuum and micro engineering and have made an important contribution to modern technologies. The oscillatory Couette flow, i.e., the flow of a gas between two parallel plates caused by the reciprocating motion of the plates, which is one of the most fundamental time-dependent flows, plays an important role in macro sensors, and thus has drawn the attention of numerous engineers and scientists [2–5]. Extensive computations based on the DSMC method have been conducted in Refs. [3,4], and the physical discussion is given in Ref. [3].

In the present study, we perform a direct numerical analysis of the oscillatory Couette flow based on the linearized Boltzmann equation for a hard sphere molecular gas. The goals of the present paper are to present a standard solution of this linear problem and to present more detailed data than that provided by the DSMC result, which would be useful in practical applications in engineering. Quite recently, Sharipov and Kalempa [5] conducted a similar study based on the linearized BGKW model of the Boltzmann equation. The result of the present study and that of Ref. [5] are compared.

2. Problem and basic equation

Consider a rarefied gas between two parallel plates placed at $X_1 = 0$ and $X_1 = L$ having a common temperature T_0, where X_i is the space rectangular coordinates. The plate at $X_1 = 0$ undergoes an in-plane reciprocating motion with the velocity $(0, V_w \cos \omega t, 0)$ (t: time). We investigate the time-dependent periodic behavior of the gas based on kinetic theory. We perform the analysis under the following assumptions: (i) the gas behavior is governed by the Boltzmann equation for a hard sphere molecular gas, (ii) the gas molecules make diffuse reflection on the boundaries, and (iii) the speed of the plate is much smaller than the sonic speed, and the equation and the boundary condition may be linearized.

The basic equation is the linearized Boltzmann equation

$$\frac{1}{k}\frac{\partial \phi}{\partial \hat{t}} + \zeta_1 \frac{\partial \phi}{\partial x_1} = \frac{1}{k}\mathscr{L}(\phi), \tag{1}$$

where $\hat{t} = t/\bar{\nu}_c^{-1}$, $x_i = X_i/L$, and ζ_i are the nondimensional time, space coordinates, and the molecular velocity, respectively, $\rho_0(2RT_0)^{-3/2}E(1+\phi)$ is the velocity distribution function, $E = \pi^{-3/2}\exp(-\zeta_i^2)$, R is the specific gas constant, ρ_0 is the mean density of the gas, $k = (\sqrt{\pi}/2)$ Kn (where $\mathrm{Kn} = \ell/L$ is the Knudsen number), $\bar{\nu}_c$

E-mail address: doi@damp.tottori-u.ac.jp

0042-207X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.vacuum.2009.10.007

and $\ell[=(8RT_0/\pi)^{1/2}\bar{\nu}_c^{-1}]$ are the mean collision frequency and the mean free path of the gas in the equilibrium state at rest, respectively, with density ρ_0 and temperature T_0. Finally, $\mathscr{L}(\varphi)$ is the linearized collision integral (see Ref. [1]).

The boundary condition is the diffuse reflection condition

$$\phi = 2\zeta_2 u_w \cos\Omega\hat{t} - 2\sqrt{\pi}\int_{\zeta_1<0}\zeta_1\phi E d\zeta_1 d\zeta_2 d\zeta_3 \quad (x_1=0, \zeta_1>0), \quad (2)$$

$$\phi = 2\sqrt{\pi}\int_{\zeta_1>0}\zeta_1\phi E d\zeta_1 d\zeta_2 d\zeta_3 \quad (x_1=1, \zeta_1<0), \quad (3)$$

where $u_w = V_w/(2RT_0)^{1/2}$ and $\Omega=\omega/\bar{\nu}_c$ is the nondimensional frequency of the oscillation.

In this problem, we can seek a solution in the following form: $\phi(x_1,\zeta_i,\hat{t}) = (\zeta_2/\zeta_\rho)\Phi(x_1,\zeta_1,\zeta_\rho,\hat{t})$, where $\zeta_\rho=(\zeta_2^2+\zeta_3^2)^{1/2}$. Furthermore, since we are interested in the periodic behavior in time, we set Φ as $\Phi(x_1,\zeta_1,\zeta_\rho,\hat{t}) = \mathrm{Re}[\exp(-i\Omega\hat{t})\Psi(x_1,\zeta_1,\zeta_\rho)]$, where $i=\sqrt{-1}$ and Re[*] denotes the real part of *. Substituting these expressions into Eqs. (1)–(3), we obtain the boundary value problem of the spatially one-dimensional Boltzmann equation, which is characterized by two parameters, namely, k and Ω.

3. Numerical analysis

The boundary value problem stated above is solved numerically using a finite difference method. The collision integral is evaluated with the aid of the numerical kernel method devised in Ref. [6]. The solution method is well established, and there is only one difficulty to be considered. In the present problem, the distribution function Ψ tends to vary rapidly in ζ_1 as Ω/k increases, and thus the numerical analysis becomes difficult for a large Ω/k. To avoid this difficulty, we deal with $\Psi-\Psi_\#$, instead of directly dealing with Ψ, where $\Psi_\#$ is a known function that is similar to the solution of the collisionless Boltzmann equation. Since the remainder $\Psi-\Psi_\#$ is considerably smooth, we can perform an accurate numerical analysis for considerably large values of Ω/k. A similar technique is used in Ref. [5].

4. Large Knudsen number and frequency

Before presenting the numerical results, we present a brief summary of the solution for large k and Ω following Refs. [3–5].

If we ignore the collision term of the Boltzmann equation (1), the solution is easily obtained, e.g., the shear stress P and the heat flow Q are given by,

$$P = 2\pi^{-1/2}J_1(-i\Omega x_1/k), \quad Q = \pi^{-1/2}[J_2(-i\Omega x_1/k) - J_0(-i\Omega x_1/k)/2], \quad (8)$$

where J_n is defined by $J_n(z) = \int_0^\infty \zeta_1^n \exp(-\zeta_1^2 - z/\zeta_1)d\zeta_1$. From the latter equation, we find $Q(0)=Q(\infty)=0$ and $|Q|$ has a simple peak at $x_1=1.12k/\Omega$. There are three typical cases in which the solution is given by Eq. (8). (i) When $k\to\infty$, keeping Ω finite (or $\Omega/k\to 0$), the solution reduces to the time-independent free molecular solution. Then, $P=\mathrm{const}=2\pi^{-1/2}J_1(0)=\pi^{-1/2}$. (ii) When $\Omega\to\infty$, keeping Ω/k finite, the solution is given by Eq. (8) with finite Ω/k. Note that this solution has a characteristic length of $Lk/\Omega \sim (2RT_0)^{1/2}/\omega$, which is different from the characteristic length $(\nu/\omega)^{1/2}$ of the Stokes layer, where ν is the kinematic viscosity of the gas. (iii) When $\Omega\to\infty$, keeping k finite (or $\Omega/k\to\infty$), the thickness of the layer expressed by Eq. (8) shrinks indefinitely.

The macroscopic quantities of the gas, i.e., the flow velocity v_i, the stress tensor p_{ij}, and the heat flow q_i, are given as follows:

$$v_2/(2RT_0)^{1/2}u_w = \mathrm{Re}\left[\exp\left(-i\Omega\hat{t}\right)U(x_1)\right], \quad v_1 = v_3 = 0, \quad (4)$$

$$p_{21}/p_0u_w = \mathrm{Re}\left[\exp\left(-i\Omega\hat{t}\right)P(x_1)\right], \quad p_{32} = p_{13} = 0, \quad (5)$$

$$p_{11} = p_{22} = p_{33} = p_0, \quad (6)$$

$$q_2/p_0(2RT_0)^{1/2}u_w = \mathrm{Re}\left[\exp\left(-i\Omega\hat{t}\right)Q(x_1)\right], \quad q_1 = q_3 = 0, \quad (7)$$

where U, P, and Q are complex valued functions of x_1 and are given by the moments of Ψ, and $p_0=R\rho_0T_0$. The other macroscopic quantities, i.e., the density ρ, the temperature T, and the pressure p, are found to be constant, i.e., $\rho=\rho_0$, $T=T_0$, and $p=p_0$.

5. Results and discussion

First, we show the profile of the macroscopic variables of the gas. As shown in Eqs. (4)–(7), the flow velocity v_2, the shear stress p_{21}, and the heat flow q_2 are expressed using the complex valued

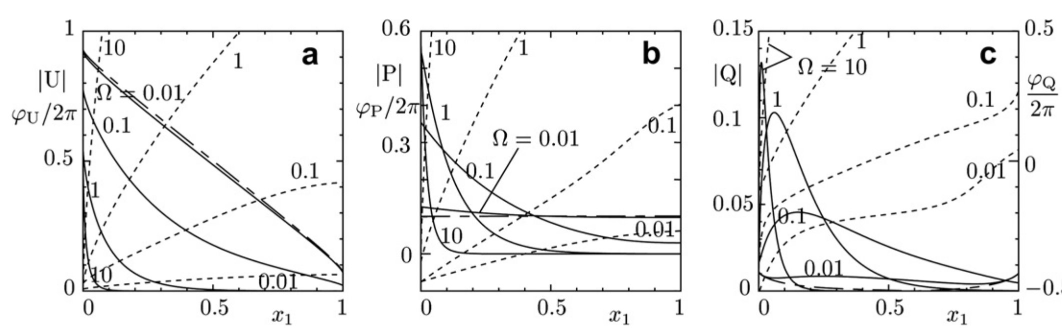

Fig. 1. The profile of the macroscopic variables I: $k=0.1$. (a) Flow velocity v_2, (b) shear stress p_{21}, and (c) heat flow q_2, where $U=|U|\exp(i\varphi_U)$, $P=|P|\exp(i\varphi_P)$, $Q=|Q|\exp(i\varphi_Q)$ [see Eqs. (4)–(7)]. ——: amplitude, - - -: phase, – – – – –. time-independent Couette flow [7].