Available online at www.sciencedirect.com

ScienceDirect

Procedia CIRP 43 (2016) 232 – 237

www.elsevier.com/locate/procedia

14th CIRP Conference on Computer Aided Tolerancing (CAT)

Theory and algorithm for planar datum establishment using constrained total least-squares

Craig M. Shakarji[a], Vijay Srinivasan[b]

[a]Physical Measurement Laboratory, National Institute of Standards and Technology, Gaithersburg, MD 20899, craig.shakarji@nist.gov
[b]Engineering Laboratory, National Institute of Standards and Technology, Gaithersburg, MD 20899, vijay.srinivasan@nist.gov

Abstract

First, we present an efficient algorithm for establishing planar datums that is based on a constrained minimization search based on the L2 norm after forming a convex surface from sampled points. Visualized by Gauss maps, we prove that the problem reduces to a minimization search where the global minimum is localized about the minimizing facet. Second, we highlight advantages of this planar datum, including the major advantage that the datum planes have full mechanical contact with the datum features in stable cases yet are automatically balanced for rocking conditions. These advantages make this definition appealing for standardization.

Peer-review under responsibility of the organizing committee of the 14th CIRP Conference on Computer Aided Tolerancing

Keywords: constrained least squares; constrained optimization; convex hull; datum; Gauss map; least squares; fitting; optimization; planar datum; singular value decomposition; total least squares

1. Introduction

In the world of Geometric Dimensioning and Tolerancing (GD&T), datums are used extensively to locate and orient tolerance zones [1-7]. Datum planes in particular are common and are established by mating planes to imperfect datum features on parts during inspection [3] (see Fig. 1). Distances and orientations on drawings and three-dimensional models are established from these datum planes, relative to which tolerance zones are located and oriented. Additional details of the importance and prevalence of datum planes in specifications are given in [8] and will not be revisited in this paper.

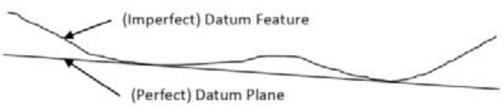

Fig. 1. Deriving a datum plane from a datum feature.

Given that datum planes are ubiquitous, it might be surprising that—short of standardization—there are several different yet reasonable approaches by which a datum plane can be established from a datum feature [9]. Furthermore, the International Organization for Standardization (ISO) and the American Society for Mechanical Engineering (ASME) are actively working to establish default datum plane definitions. In [10] we introduced a definition for a planar datum that naturally combines a correspondence to physical, surface plate mating (i.e., "high points") but with automatic balancing in the case of unstable, rocking conditions. The datum plane definition is based on a constrained total least-squares criterion (abbreviated here as L2C), which is explored in this paper. This should not be confused with an unconstrained total least-squares fit that is shifted out of the material.

Given a set of points sampled on a datum feature, the two major steps in establishing the L2C datum plane are as follows:

1) Compute the "lower" convex envelope of those points. This is the portion of the convex hull that lies on the nonmaterial side of the datum feature. In 3D, this convex envelope consists of a union of non-overlapping triangles, while in 2D it is a union of line segments creating a piecewise linear curve.
2) Find the plane, constrained to lie on the nonmaterial side of the computed convex surface that minimizes the integral of squared distances from that surface, namely $\int_S d^2(\boldsymbol{p}, P)ds$, where S is the convex surface and d is the distance from a point $\boldsymbol{p}$ on the surface to the

Peer-review under responsibility of the organizing committee of the 14th CIRP Conference on Computer Aided Tolerancing
doi:10.1016/j.procir.2016.02.104

plane, P. If P contains $\boldsymbol{x}$ and has normal $\boldsymbol{a}$, then $d = \boldsymbol{a} \cdot (\boldsymbol{p} - \boldsymbol{x})$.

Concentrating on the second step, we find the need to integrate over a set of triangles (or line segments in 2D). For each triangle (or line segment) this integral can be replaced by the Simpson's rule approximation (see Fig. 2) [11] (which we will see is actually exact in our case).

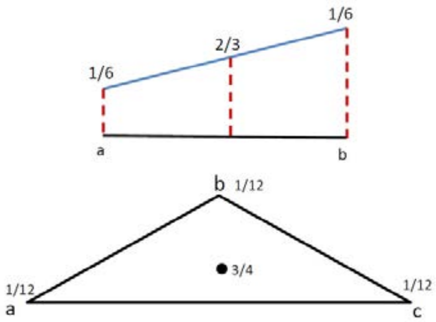

Fig. 2. The locations and weights for function evaluations for numerical integration using Simpson's rule over an interval and triangle.

Simpson's rule for integrating over an interval (or triangle for the 3D case) depends only on the weighted values of the function at the endpoints (or vertices in 3D) and at the centroid. Over an interval, Simpson's rule is given by:

$$\int_a^b f(x)dx \approx (b-a)\left(\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right),$$

and for integrating over a triangle, T, as shown in Fig. 2,

$$\int_T f(\boldsymbol{s})dT \approx$$

$$\text{Area}(T)\left(\frac{1}{12}f(a) + \frac{1}{12}f(b) + \frac{1}{12}f(c) + \frac{3}{4}f\left(\frac{a+b+c}{3}\right)\right).$$

Because Simpson's rule [11] is exact for functions of degree 2 (our case), we note that in the two formulas just above, these are exact calculations of the integrals and not mere approximations. The framing of this problem as a weighted sum-of-squares now allows us to solve the objective function as a singular value decomposition (SVD) problem. See [12] for a general treatment of using the SVD as a method for minimizing the total least-squares problem, and [13] for an application of it applied to planar fitting with weighted points (essential to be physically correct), which is our case here.

For the 3D case, let a S be a lower convex surface be made up of N triangles, T_1, T_2, …,T_N, where T_i has vertices (x_{iA}, y_{iA}, z_{iA}), (x_{iB}, y_{iB}, z_{iB}), and (x_{iC}, y_{iC}, z_{iC}) and where each triangle has centroid $(\bar{x}_i, \bar{y}_i, \bar{z}_i)$ and area A_i. If P is a candidate plane and, for each triangle, d_{iA}, d_{iB}, d_{iC} are the distances between P and the vertices and $\bar{d}_i$ is the distance from P to the triangle's centroid. Then, the L2C objective function to be minimized is:

$$\sum_{i=1}^{N} A_i\left(\frac{d_{iA}^2}{12} + \frac{d_{iB}^2}{12} + \frac{d_{iC}^2}{12} + \frac{3\bar{d}_i^2}{4}\right). \quad (1)$$

For the 2D case, where the convex surface is comprised of $N-1$ line segments, each having length L_i , endpoints (x_i, y_i), and (x_{i+1}, y_{i+1}), d_i being the distance from P to (x_i, y_i), and $\bar{d}_i$ is the distance from P to the line segment's midpoint, we then have the objective function being

$$\sum_{i=1}^{N-1} L_i\left(\frac{d_i^2}{6} + \frac{d_{i+1}^2}{6} + \frac{2\bar{d}_i^2}{3}\right). \quad (2)$$

In [10] we proved that the (2D) objective function for any candidate plane P is given by the elegant, efficient formula:

$$\sigma_1^2\text{Cos}^2\theta + \sigma_2^2\text{Sin}^2\theta + Ld_c^2, \quad (3a)$$

or equivalently

$$\sigma_1^2 a^2 + \sigma_2^2 b^2 + Ld_c^2, \quad (3b)$$

where (see Fig. 3) d_c is the distance from the plane P to the centroid, σ_1 and σ_2 are the singular values from the SVD of the matrix $\boldsymbol{M}$ below, and θ represents the angle P makes with the singular vector corresponding to the smallest singular value, σ_1 . Eqs. (3a) and (3b) are equivalent, where $(a,b) = (\text{Cos}\theta, \text{Sin}\theta)$ is the unit normal to the candidate plane when expressed as the dot product of that normal with each of the two singular vectors (e.g., a is the dot product of the unit normal to the plane with the first singular vector). The $3N \times 2$ matrix, $\boldsymbol{M}$, that is used in the SVD comes from the elements the Simpson's rule approximation (see [10] for more detail), repeated for each of the N line segments:

$$\boldsymbol{M} = \sqrt{\frac{1}{6}}\begin{bmatrix} \sqrt{L_1}(x_1) & \sqrt{L_1}(y_1) \\ 2\sqrt{L_1}\left(\frac{x_1+x_2}{2}\right) & 2\sqrt{L_1}\left(\frac{y_1+y_2}{2}\right) \\ \sqrt{L_1}(x_2) & \sqrt{L_1}(y_2) \\ \vdots & \vdots \\ \sqrt{L_N}(x_N) & \sqrt{L_N}(y_N) \\ 2\sqrt{L_N}\left(\frac{x_N+x_{N+1}}{2}\right) & 2\sqrt{L_N}\left(\frac{y_N+y_{N+1}}{2}\right) \\ \sqrt{L_N}(x_{N+1}) & \sqrt{L_N}(x_{N+1}) \end{bmatrix}$$

(The construction of $\boldsymbol{M}$ is done with the data translated so the centroid is at the origin. This translation is not shown explicitly in the matrix for reasons of space.)

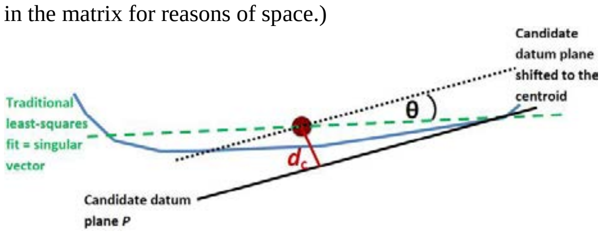

Fig. 3. The objective function for any candidate datum can be found simply by knowing the angle θ and distance d_c and using Eq. (3).

Using Eq. (3) to compute the objective function means that the SVD has to be computed only once, and its result can be applied to any given candidate datum plane. This makes for a much more efficient minimization algorithm.

What is fascinating about Eq. (3) is that the two terms on the left are exactly the objective function used in a traditional least-squares minimization while the term on the right is the objective function in a constrained L_1 fit [14, 15]. We will see that the objective function indeed does manifest itself as having the balancing property of the unconstrained least-squares and the full mechanical contact of the constrained L_1 definition, which is what is desired.

This can extend to 3D as well, since we showed that there is an extension of Simpson's rule that applies to integration over a triangular region. For the 3D case, the objective function for any candidate plane P is given by the efficient formula: