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Analytical solution to the 1D Lemaitre's isotropic damage model and plane stress projected implicit integration procedure

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ABSTRACT

In the present paper, for the first time in literature an exact analytical solution to Lemaitre's isotropic damage model is developed for the special case of uniaxial tensile testing. This is achieved by taking advantage of a convenient formulation of the isotropic hardening function, which allows obtaining an integral relationship between total strain and effective stress. By means of the generalized binomial theorem, an expression in terms of infinite series is subsequently derived. The solution is found to simplify considerably existing techniques for material parameters identification based on optimization, as all issues associated with classical numerical solution procedures of the constitutive equations are eliminated. In addition, an implicit implementation of the plane stress projected version of Lemaitre's model is discussed, showing that the resulting algebraic system can be reduced to a single non-linear equation. The accuracy of the proposed integration scheme is then verified by means of the presented 1D analytical solution. Finally, a closed-form expression for the consistent tangent modulus taking damage evolution into account is given, and its impact on the convergence rate is analyzed.

1. Introduction

Lemaitre's model [1] is today one of the most widely used techniques for modeling damage evolution in ductile materials. Despite several improvements have been proposed in order to account for additional effects like micro-cracks closure under compressive stresses [2] and anisotropy [3,4], the original isotropic formulation limited to isotropic hardening is still often employed due to its simplicity and the relatively low number of material parameters involved. The latter can be easily determined from knowledge of the damage evolution history in loaded specimens, obtainable by means of well-established experimental methods [5]. Nevertheless, it might be that such information is either not available or too expensive to obtain for the specific material at hand. Under these circumstances, it is common practice to identify the material constants on the basis of an optimization analysis aiming at minimizing the error between the predicted numerical results and a reference uniaxial tensile curve [6]. More recently, methodologies which offer improved accuracy by combining the classical experimental stress–strain relationship with other observables have also been proposed [7,8]. However, the use of all such procedures in combination with a numerical resolution of the mathematical model exhibits three main disadvantages. First

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Nomenclature

s_{ij}	Deviatoric part of the stress tensor
δ_{ij}	Kronecker delta
ε_{ij}^{tot}, ε_{ij}^{e}, ε_{ij}^{p}	Total/elastic/plastic strain tensor
σ_{ij}	Stress tensor
$\tilde{\sigma}_{ij}$	Effective stress tensor
σ_e	Equivalent Von Mises stress
σ_y, σ_y^0	Actual/initial yield stress
E	Young's modulus
D	Damage variable
R_v	Triaxiality function
S, s	Lemaitre's damage evolution parameters
Y	Energy release rate
Φ	Yield function
k, n	Isotropic hardening parameters
p	Equivalent Von Mises plastic strain
p_{crit}	Critical effective plastic strain for damage evolution
r	Hardening variable
λ	Plastic multiplier
ν	Poisson's ratio
$(\dot{\ })$	Increment operator
$\Delta()$	Finite variation operator

of all, it requires the use of an optimization algorithm coupled with a numerical tool for solving the underlying system of differential equations. Secondly, the numerical solution procedure in itself is complicated by the "softening" behavior of the damaged material at sufficiently large strains. Thirdly, the produced values for the material parameters are affected, in addition to the experimental & optimization uncertainty, by the error associated with the numerical discretization adopted. With the aim of overcoming all the above-mentioned aspects, an analytical solution to the isotropic Lemaitre's model for the specific case of uniaxial tensile testing is developed in the present paper. Quite surprisingly, no previous works on the subject appear to have been published in literature, despite the intrinsic value that analytical solutions possess, especially in relation to numerical implementations assessment.

Lemaitre's model has also proved effective in predicting damage evolution in situations of plane stress, as recently confirmed by geometric transferability investigations carried out on flat specimens of Ti-6Al-4 V alloy [9]. These characteristic loading conditions are frequently encountered in industrial processes where material degradation becomes a critical factor, as discussed in [10] for the case of sheet metal forming. If a traditional displacement-based implicit finite element code with global full Newton–Raphson iterations is used for the related numerical analysis, a natural complication arises in ensuring the out-of-plane components of the stress tensor to be zero at the end of each load increment. This issue is thoroughly discussed in the context of general elasto-plasticity in [11], where three different techniques are suggested to cope with the problem: (A) direct inclusion of the plane stress constraint at the Gauss point level, (B) addition of a plane stress constraint at the global structural level, (C) use of plane stress projected constitutive equations at the Gauss point level. The first two solutions are usually easier to implement, but at the price of higher computational time; an example of implicit constitutive discretization of Lemaitre's model with kinematic hardening using approach (A) is given in [12]. As a consequence, approach (C) is to be preferred whenever possible, as it leads to more efficient computational procedures due to the fact that only the relevant in-plane stress and strain components are considered [13]. This is also the approach adopted in the second part of the present paper, where it is shown that implicit integration of the plane stress projected Lemaitre's model with isotropic hardening using the elastic predictor-return mapping scheme can be reduced to a single non-linear equation. Furthermore, a closed-form expression for the related consistent tangent modulus is proposed, which is found to improve significantly the convergence rate.

2. Lemaitre's isotropic damage model with isotropic hardening

The fundamental equations characterizing Lemaitre's isotropic damage model with isotropic hardening are reported here in Cartesian components [6]:

(a) additive strain decomposition:

$$\varepsilon_{ij}^{tot} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p}, \tag{1}$$