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The numerical simulation of the tempered fractional Black–Scholes equation for European double barrier option☆

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ABSTRACT

In recent years, the Finite Moment Log Stable(FMLS), KoBoL and CGMY models, which follow a jump process or a Lévy process, have become the most popular modeling frameworks in the financial field because they can capture some of the important characteristics in the dynamic process of stock price changes, such as large movements or jumps over small time steps. In this paper, we consider the numerical simulation of these three models. We construct a discrete implicit numerical scheme with second order accuracy, and provide a stability and convergence analysis of the numerical scheme. Furthermore, a fast bi-conjugate gradient stabilized method (FBi-CGSTAB) is used to reduce the storage space from $O(M^2)$ to $O(M)$ and the computational cost from $O(M^3)$ to $O(M\log M)$ per iteration, where M is the number of space grid points. Some numerical examples are chosen in order to demonstrate the accuracy and efficiency of the proposed method and technique. Finally, as an application, we use the above numerical technique to price a European double-knock-out barrier option, and then the characteristics of the three fractional Black–Scholes (B–S) models are analyzed through comparison with the classical B–S model.

1. Introduction

An option is one of the most important and popular financial derivatives in the financial market and during the last decades the valuation of option contracts has been a popular topic. There are various types of mathematical models for pricing different kinds of options. In the 1970s, Black and Scholes [1] and Merton [2] developed an original option pricing formula governing the price of the option over time, which is known as the classical Black–Scholes (B-S) (or Black–Scholes–Merton) options pricing model. Since then, the B–S model has attracted interest in the financial field because of its simplicity and clarity in obtaining the price of the option. However, it is well known that the classical B–S model was established under some strict assumptions, such as frictionless, price changing smoothly and option be exercised at maturity. Therefore, to weaken these assumptions some improved models or more general models have been proposed. These include the model with transaction costs [3,4], the jump-diffusion model [5,6] and the stochastic volatility model [7,8]. Moreover, empirical evidence has suggested that one of the significant shortcomings of the classical B–S model is that Gaussian shocks

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underestimate the probability that stock prices exhibit significant movements or jumps over small time steps in a financial market. To overcome this problem, the Lévy processes are introduced to replace the standard Brownian motions. Lévy processes can not only exhibit log fat tails but also capture the features of volatility skew or smile that is absent in the classical B–S models. In early 1960s, Mandelbrot [9] observed that the relative stock price changes exhibited a phenomenon of a long-tailed distribution and proposed an exponential non-normal Lévy process for modeling this phenomenon. Furthermore, he deduced that an α-stable Lévy motion with index $\alpha < 2$ should be used instead of the standard geometric Brownian motion to simulate these phenomena. Koponen [10] and Boyarchenko and Levendorskiĭ [11] proposed the application of modified Lévy-α-stable processes in modeling the dynamics of securities, which are known as KoBoL processes in the mathematical finance literature. Carr, Geman, Madan and Yor [12] put forward a CGMY process which allowed for jump components displaying finite and infinite activity and variation. FMLS process was presented by Carr and Wu [13], which can capture the highly skewed feature of the implied density for log returns. These three Lévy processes, which can capture the most important characteristics of the dynamics of stock prices, have gradually become the very favored models in financial field in recent years.

Fractional derivatives are quasi-differential operators, which provide useful tools for a description of memory and hereditary properties and are closely related to Lévy processes. The differential equations with fractional derivatives are powerful tools for studying fractal geometry and fractal dynamics [14–16] and have been used in a variety fields to model anomalous diffusion phenomena or α-stable Lévy processes [15,17–21]. Recently, with the discovery of the fractal structure in financial markets, the fractional models have been introduced more and more into the financial field. Wyss [22] presented a time fractional B-S model to price a European call option. Cartea and del-Castillo-Negrete [23] derived the FMLS, KoBol and CGMY models for pricing exotic options in markets with jumps. By using the fractional order Taylor's formula, Jumarie [24,25] derived the time and space fractional B-S models for stock exchange dynamics and gave an optimal fractional Merton's portfolio. Li [26] proposed a time fractional Black–Scholes–Merton model according to the connection of the fractal structure and an options diffusion process. The bi-fractional Black–Scholes–Merton model of option pricing was proposed by Liang et al. [27]. Recently, assuming that the two underlying assets follow two independent geometric Lévy processes, Wen Chen derived European and American fractional B-S models for two-asset options [28].

As fractional models are more and more widely used in the financial field, the problem of how to solve them has attracted the interest of more and more researchers. Wyss [22] gave the complete solution of a time fractional B-S equation by Laplace and Mellin transforms. Jumarie [25] presented the solutions of time-fractional and space-fractional B–S models by Fourier transforms and Mittag-Leffler function. Liang, et al. [27] obtained an analytical solution of a bi-fractional Black–Scholes–Merton model with the help of the Laplace transform technique. Kumar et al. [29] discussed an analytical solution of a time fractional B–S European option pricing equation using the same technique. The homotopy perturbation method and Sumudu transform were used to consider the solution of the time fractional B–S European option pricing equations by Elbeleze et al. [30]. Chen et al. [31] considered the analytical solution of the FMLS model by using a Fourier integral transform and Fox functions. According to the literature review above, it appears that the favored methods used to consider the analytical solution of the fractional B–S models are via integral transform methods. The solutions obtained by means of these methods usually take the form of a convolution of some special functions, which make it difficult to compute. Therefore, studying the numerical approximate solutions of these models appears to be a very practical and important research objective.

It is well known that a great number of recent publications consider the numerical simulation of differential equations with integer order derivatives arising from pricing the European or American option. However, research focused on the corresponding differential equations with fractional order derivatives are relatively limited, and numerical approximations of these equations are still at an early stage of development. Cartea and del-Castillo-Negrete [23] solved the FMLS model numerically by using the shifted Grüwald–Letnik scheme and backward difference techniques. Then the authors applied the proposed numerical techniques to price exotic options, in particular barrier options. Marom and Momoniat [32] presented a comparison of numerical solutions of the FMLS , KoBol and CGMY models and obtained the conditions for the convergence of each of these models. But the stability and convergence of the proposed numerical schemes were not considered. Li [26] gave a difference scheme with first order accuracy for a space fractional B–S model. By using a wavelet technique, G. Hariharan [33] solved a time fractional B–S European option pricing problem. Chen [28] proposed a second order finite difference method for the one-dimensional FMLS model governing the valuation of European options and a power penalty method for a space fractional differential linear complementarity problem arising in the valuation of American options on a single asset, then extended the above methods to the corresponding two-dimensional models arising in the valuation of European and American options on two assets. The author also gave the detailed numerical analysis of the methods. Song and Wang [34] solved a time fractional B–S option pricing model numerically by finite difference method. Recently, Zhang, et al. [35] considered the same model as [34] by a θ-difference scheme with first order accuracy in time and second order accuracy in space.

Multiplying by an exponential factor, the usual fractional calculus leads to the tempered fractional calculus. The tempered fractional operators are commonly used in truncated exponential power law description [10]. The tempered fractional diffusion equations with the tempered fractional derivatives can be used to govern the limits of random walk models with an exponentially tempered power law jump distribution [36,37]. The phenomena of semi-heavy tails, that are commonly observed in finance, arise by limited tempered stable probability densities [12,23]. For the numerical simulation of the usual fractional advection–diffusion equations there are many literatures considered it [17,38–44]. But for the fractional