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Geometric degree of nonconservativity: Set of solutions for the linear case and extension to the differentiable non-linear case

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ABSTRACT

This paper deals with nonconservative mechanical systems as those subjected to nonconservative positional forces and leading to non-symmetric tangential stiffness matrices. In a previous work, the geometric degree of nonconservativity of such systems, defined as the minimal number ℓ of kinematic constraints necessary to convert the initial system into a conservative one is found to be, in the linear framework, the half of the rank of the skew-symmetric part of the stiffness matrix. In the present paper, news results are reached. First, a more efficient solution of the initial linear problem is proposed. Second, always in the linear framework, the issue of describing the set of all corresponding kinematic constraints is given and reduced to the one of finding all the Lagrangian planes of a symplectic space. Third, the extension to the local non-linear case is solved. A four degree of freedom system exhibiting a maximal geometric degree of nonconservativity ($s = 2$) is used to illustrate our results. The issue of the global non-linear problem is not tackled. Throughout the paper, the issue of the effectiveness of the solution is systematically addressed.

Introduction

Nonconservative elastic mechanical systems exhibit several paradoxical mechanical behaviors. Destabilizing effect by additional friction is certainly the most famous paradox of these mechanical systems and has been deeply investigated (see [1–3] for example). One less reported paradoxical effect is the destabilizing effect by additional kinematical constraints. J.J. Thompson mentioned this effect in [4] but, to the best of our knowledge, this paradoxical effect had never been systematically investigated before recently. This paradoxical effect led to the so-called kinematical structural stability (ki.s.s.) issue: when and how is it possible to destabilize by adding kinematical constraint(s) a given stable system?

During the last five years, in a sequence of papers ([5–9]), we elucidated this kinematical structural stability (ki.s.s.) issue for the linear divergence stability of both conservative and nonconservative elastic systems as well. A big part of these works are also related to the so-called second order work criterion introduced by Hill in the framework of plasticity in 1958 ([10])

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and independently introduced and used in the framework of elastic nonconservative systems in 2004 ([11]). The main result involves the symmetric part K_s of the stiffness matrix and the magnitude of the load parameter as well but it does not depend on the number of the additional kinematic constraints.

By duality to the ki.s.s. issue, we investigated in [12] the issue to convert by (judicious) additional kinematic constraints a nonconservative system Σ into a conservative one. This issue leads to the concept of geometric degree d of nonconservativity of Σ. Calculations show that $d = s$ is the half s of the rank r of the skew-symmetric part $K_a(p)$ (that is always even $r = 2s$). In a second stage, a building of the judicious additional kinematic constraints $C_1, \dots, C_s \in (\mathbb{R}^n)^s$ has been proposed thanks to the eigenspaces $E_{-\lambda_i^2}, i = 1, \dots, s$ of the symmetric matrix $K_a^2(p)$ whose the eigenvalues $-\lambda_1^2, \dots, -\lambda_s^2$ are all double: each C_i may be chosen in each distinct $E_{-\lambda_i^2}$. It is worth noting that, for both issues, the mechanical system Σ is approximated by its linear first order approximation at a given equilibrium configuration q_e. That means that Σ is described by the mass matrix M and the stiffness matrix K. If p is a load parameter, then $K = K(p)$. The non-symmetry of $K(p)$ (namely $K \neq K_s$ or $K_a \neq 0$) is then the signature of the non-conservative nature of the mechanical system Σ. In our previous works, the source of the nonconservativity lies in external forces like follower forces acting on elastic system. Hypoelasticity may also be another mechanical framework leading to a similar mathematical problem. There exists a broad literature covering hypoelasticity (see for example [13–16]).

In this paper we are concerned by finding the complete solution of the linear case and by the generalization and the extension to the non-linear differentiable case about to the latter issue. We then use the language of analytic mechanics. In a first time, we reinvestigate the linear case by using the language of exterior p-forms and especially exterior 1- and 2-forms. That allow us to more deeply highlight the issue of effectiveness of the calculation of the suitable kinematic constraints converting the system into a conservative one. That also allow us to investigate the issue of building the set of all the solutions and to illustrate the geometrical meaning of these solutions. To do it, the language of symplectic geometry is systematically used. That also suggests the good way for tackling the non-linear case.

Thus, in a second time, we tackle the non-linear problem with appropriate notations and especially thanks to the language of differential p-forms. We accurately focus on the link with the linear case. In a third step, the solution is proposed by extending to the nonlinear case the concept of geometric degree of nonconservativity and yielding a geometric meaning to the corresponding non-linear constraints. In the last part, the issue of the calculation of the appropriate non-linear constraints is investigated. The problem of a global solution in relationship with the topology of the configuration manifold is only evoked by just setting the convenient geometric framework of vector bundles. A four degree of freedom system called the Bigoni system (see [12,17]) is continuously used throughout the paper to illustrate the general results.

1. The linear case

In what follows we refer to [12]. We only recall that for the linear framework, dynamic equation of the unconstrained system Σ read:

$$M\ddot{X} + KX = 0, \tag{1}$$

with K any (namely non-symmetric) matrix and M symmetric positive definite. K is the stiffness matrix of the system and M its mass matrix. Because of the nonconservativity of the positional forces acting on σ, K is any. The minimum number of kinematic constraints allowing to convert the system into a conservative one (with a corresponding symmetric stiffness matrix) is the geometric degree of nonconservativity of Σ. (1) is deduced from the Lagrange equation by the usual process of linearization about an equilibrium configuration.

1.1. *Effectiveness of the solution proposed in [12]*

In introduction, we already recalled the algebraic meaning of the geometric index or degree of nonconservativity: this the half s of the rank $r = 2s$ of K_a and the distinct constraints, viewed as vectors of $\mathbb{R}^n$, can be chosen in the s distinct eigenspaces $E_{-\lambda_i^2}, i = 1, \dots, s$ of K_a^2. We now question the effectiveness of the building of the constraints as proposed in [12]. To do it, we use the spectral theorem for K_a^2. What does mean the effectiveness for the spectral theorem? The usual proof is done by induction on the dimension of the space. For initializing the induction reasoning, the D'Alembert Gauss theorem is used for finding an eigenvalue of the characteristic polynomial of K_a^2 and this theorem is not effective in the sense where only a numerical method may lead to (an approximation of) the eigenvalues. So, with these tools, the solution of the linear case itself is not effective. Remark however that the constraints are also the critical points of the Rayleigh quotient R associated with K_a^2 and that only the eigenspaces are interesting and not the eigenvalues $-\lambda_i^2, i = 1, \dots, s$. The use of Rayleigh quotient is then especially relevant and the constraints may be evaluated by successive minimizations of $R(X) = -\frac{X^T K_a^2 X}{X^T X}$. By Minimax theorem, the constraints are also the solutions of

$$\min_{\dim F = k} \max_{X \in F \setminus \{0\}} R(X),$$

for $k = 1, \dots, n$ avoiding by this way the use of D'alembert–Gauss theorem. However, this minimization process gives no analytic explicit result.