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Bio-inspired computing platform for reliable solution of Bratu-type equations arising in the modeling of electrically conducting solids

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A B S T R A C T

In this study, a bio-inspired computing approach is developed to solve Bratu-type equations arising in modeling of electrically conducting solids and various other physical phenomena. We employ feed-forward artificial neural networks (ANN) optimized with genetic algorithm (GA) and the active-set method (ASM). The mathematical formulation consists of an ANN with an unsupervised error, which is minimized by tuning weights of the network. The evolutionary technique based on GAs is used as a tool for global search of the weights in conjunction with the ASM for rapid local convergence. The designed methodology is applied to solve a number of initial and boundary value problems based on Bratu equations. Monte Carlo simulations and their statistical analyses are used to validate accuracy, convergence and effectiveness of the scheme. Comparison of results is made with exact solutions, the fully explicit Runge–Kutta numerical method, and other reported solutions of analytical and numerical solvers to establish correctness of the designed scheme.

1. Introduction

In this study we develop a bio-inspired computing approach to solve a second order nonlinear differential equation given as:

$$\frac{d^2y(t)}{dt^2} = \lambda(t)e^{\mu(t)\,y(t)}, \tag{1}$$

defined on the close interval [0, 1] subject to the following boundary conditions:

$$y(0) = y(1) = 0,$$

where $\lambda(t)$ and $\mu(t)$ are some known continuous functions of *t*. The problem given above arises in modeling of electrically conducting solids [1]. $\mu = 1$ in the above equation arises in the analysis of Joule losses in electrically conducting solids, λ

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represents square of the constant current and e^y models the temperature dependent resistance. In the case of fractional heating λ represents square of the constant shear stress and e^y models temperature dependent fluidity. More details about this model can be found in [1] for the interested reader.

Bratu's equation is a special form of Eq. (1) where the functions $\lambda(t)$ and $\mu(t)$ are constants. Its general form is given as:

$$\frac{d^2y(t)}{dt^2} + \lambda e^{\mu\, y(t)} = 0, \quad 0 \le t \le 1. \tag{2}$$

The associated initial and boundary conditions, respectively, are given as:

$$y(0) = \frac{dy(0)}{dt} = 0 \, \text{and} \, y(0) = y(1) = 0,$$

where μ is taken as +1 or -1, $y(t)$ is the solution of the equation, and λ is a real parameter. This two point boundary value problem is a special case in the modeling of electrically conducting solids, and also occurs in diffusion theory [1,2]. Eq. (2) with initial and boundary conditions also arises in many physical models such as the fuel ignition model in combustion theory, model of thermal reaction processes in chemical reaction theory, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity concerning the Chandrasekhar model, radiative heat transfer, and nanotechnology [2–6]. The Bratu problem came to light with Bratu's paper published in 1914 [3]. In honor of Gelfand and the French mathematician Liouville [4], it is also known as the "Liouville–Gelfand" or "Liouville–Gelfand–Bratu" problem. Jacobsen and Schmitt have provided an excellent summary of the significance and history of Bratu-type equations [5]. In recent years, this problem has become a popular benchmark to test the accuracy of various numerical solvers [7,8]; stochastic approaches based on optimized artificial neural networks (ANNs) however have been less utilized for the solution of these boundary value problems (BVPs).

In the field of optimization, genetic algorithms are considered to be best for global search and are widely used as efficient optimizers in various applications such as design of wireless vibration control for a beam by using photostrictive actuators [9], solution of an inverse heat conduction problem [10], and parameter estimation in robotics based on the Takagi–Sugeno fuzzy model [11] etc. ANNs optimized with evolutionary techniques and swarm intelligence based algorithms have been used to solve a variety of initial and BVPs associated with linear and non-linear differential equations [12–14]. These methodologies have been extended to solve linear and non-linear fractional differential equations, including Riccati differential equations of arbitrary order and the Bagley–Torvik fractional system [15–17]. Differential equations possessing strong non-linearity, such as the first Painlevé transcendent, Troesch problem with stiff and non-stiff conditions, nonlinear oscillators, Pantograph Functional Differential Equation, nonlinear singular Flierl–Petviashivili equation for unbounded domain, one and two dimensional Bratu's equations are also illustrative applications of such solvers [18–27]. Recently, these methodologies have been used to solve nonlinear MHD Jeffery–Hamel flow problems effectively [28–30], problems arising in electromagnetic theory [31], nanotechnology problems based on carbon nanotubes [32], steady Thin Film Flow of Johnson–Segalman fluid on a vertical cylinder for drainage problem [33] and solid fuel ignition model based on Bratu's equations [34]. This motivates the authors to explore solution of relatively complex initial value problems (IVPs) and BVPs of equations arising in modeling of various physical phenomena. The aim of this study is to provide an alternate, accurate, reliable and robust framework to solve one-dimensional IVPs and BVPs of the equation by exploiting bio-inspired computing techniques based on ANNs, GAs and ASMs.

In this paper, we exploit the strength of ANNs for modeling the one-dimensional Bratu problem. The correctness of the model is subject to proper adjustment of weights of the ANN. For this an evolutionary computing technique based on GAs is applied for training weights of the network. The best learned weights from the GA are provided to the ASM as a starting point for further fine tuning by rapid local optimization. A large number of independent runs of the designed scheme are performed in order to validate the effectiveness and reliability of the proposed method. Comparison of the results is made with the exact solution, numerical solutions based on Runge–Kutta methods, and solutions obtained by solvers including the B-spline technique (BST) [35], Adomian decomposition method (ADM) [36,37], Laplace transform decomposition method (LTDM) [38], non-polynomial spline method (NSM) [39], and the Lie-group shooting method (LGSM) [40].

2. Neural network model of Bratu's equation

In this section an ANN mathematical model for one-dimensional Bratu-type equations is developed, along with the formulation of the fitness function.

Using Artificial Neural Network (ANN) methodology, the solution $y(t)$ of the equation along with its nth order derivative $d^ny(t)/dt^n$, are formulated by the following continuous mappings [34,41,42]:

$$\hat{y}(t) = \sum_{i=1}^{k} \alpha_i f(w_i t + \beta_i), \tag{3}$$

$$\frac{d^n}{dt^n}\hat{y}(t) = \sum_{i=1}^{k} \alpha_i \frac{d^n}{dt^n} f(w_i t + \beta_i), \tag{4}$$