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A linear finite difference scheme for generalized time fractional Burgers equation☆

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ABSTRACT

This paper is concerned with the numerical solutions of the generalized time fractional burgers equation. We propose a linear implicit finite difference scheme for solving the equation. Iterative methods become dispensable. As a result, the computational cost can be significantly reduced compare to the usual implicit finite difference schemes. Meanwhile, the finite difference method is proved to be unconditional globally stable and convergent. Numerical examples are shown to demonstrate the accuracy and stability of the method.

1. Introduction

In this study, we present a linear implicit finite difference scheme for the numerical solutions of the following generalized time fractional Burgers equation

$$ {}_0^C D_t^\alpha u(x,t) = d\frac{\partial^2 u(x,t)}{\partial x^2} - u^p \frac{\partial u(x,t)}{\partial x}, \quad 0 \le x \le L, \quad 0 \le t \le T, \tag{1.1}$$

$$ u(0,t) = u(L,t) = 0, \quad u(x,0) = u_0(x), \tag{1.2}$$

where $0 < \alpha < 1$, d, p are all positive constants, the symbols ${}_0^C D_t^\alpha$ is Caputo's fractional derivative operator defined by

$$ {}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^\alpha} ds, \tag{1.3}$$

where $\Gamma(\cdot)$ denotes the usual gamma function.

The fractional Burgers equation was used to describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the cumulative effect of the wall friction through the boundary layer (cf. [1,2]). The same form can also be found in other models such as shallow-water waves and waves in bubbly liquids. We refer to [3–7] for an incomplete list of references on the applications associated with of the

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fractional Burgers equation. So far, there are considerable methods for solving the fractional burgers-type equations. In 2006, the adomian decomposition method was used to solve the fractional Burgers equations in [8]. After that, the variational iteration method [2] and homotopy analysis method [9,10] were directly extended to derive explicit and numerical solutions of the fractional Burgers-type equations, respectively. The solutions obtained by the above methods were calculated in the form of convergent series. Recently, a parametric spline based method was developed to approximate the solution to the problem in [11]. The truncation error of the method and stability of the method were investigated. For a more detailed description of this subject, we refer the readers to the papers [12–18], the books [19,20] and the references therein.

Besides, the finite difference scheme is also an effective numerical method for solving variety kinds of fractional differential equations (see e.g. [21–30]). The method are widely accepted because of its simplicity and intuition. In [31] and [32], the stability and convergence of several kinds of finite difference schemes were discussed for solving the fractional differential equation. However, as it is pointed in their study, in the case of explicit method, the proper small step sizes should be chosen to keep the numerical method stable. In the case of implicit method, although the method may be unconditional stable in most cases, one obtain a new system of nonlinear algebraic equations at each time step even for a fixed parameter p (please also see Section 4 in this study). The consequent iterative schemes for computing the unknown values make the numerical method prohibitively expensive to use. Generally speaking, the computational complexity or cost increases as the parameter p in the advection term of (1.1) becomes larger and larger. Although a lot of effort is being spent on improving these weaknesses, e.g. split Newton iterative algorithm [33], inexact Newton iterative algorithm [34], the inherent problem has not been solved.

A major thrust of the paper is to develop the efficient and effective numerical method, which can overcome the mentioned difficulties for the generalized time fractional burgers equations. First of all, a linear implicit finite difference scheme is proposed for solving the generalized time fractional burgers equation. We show that the given finite difference method is unconditional globally stable and convergent. Besides, the method is linear. Iterative methods become dispensable when we solve the problem with different parameters p. The computational cost can be significantly reduced compare to the usual implicit finite difference schemes. Finally, numerical examples are shown to demonstrate the accuracy and stability of the method.

The rest of the paper is organized as follows. Section 2 describes in detail the derivation of the finite difference scheme. Section 3 show the stability and convergence of the finite difference scheme. Section 4 is developed to compare our numerical scheme with the usual one. Section 5 shows experimental studies for verifying the proposed results. Finally, conclusions and discussions for this paper are summarized in Section 6.

2. The derivation of the finite difference scheme

In this section, we will describe in detail the derivation of the numerical method for the generalized time fractional Burgers equation.

Let $\tau = \frac{T}{N}$ and $h = \frac{L}{M}$ be the temporal and spatial step sizes, respectively, where M and N are given positive integers. Denote $t_n = n\tau$ $(0 \le n \le N)$, $x_j = jh$, $(0 \le j \le M)$, $\Omega_\tau = \{t_n | 0 \le n \le N\}$ and $\Omega_h = \{x_j | 0 \le n \le N\}$. Let u_j^n denote the numerical approximation of $u(x_j, t_n)$ and $\mathcal{V} = \{u_j^n | 0 \le j \le M, 0 \le n \le N\}$ be grid function space defined on $\Omega_h \times \Omega_\tau$. For any grid function $u \in \mathcal{V}$, we will provide the notations which are necessary for the understanding of subsequent numerical schemes.

$$(u_j^n)_x = \frac{u_{j+1}^n - u_j^n}{h}, \quad (u_j^n)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h}, \quad (u_j^n)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h},$$

$$(u^n, v^n) = h\sum_{j=1}^{M-1} u_j^n v_j^n, \quad \|u^n\|^2 = (u^n, u^n), \quad \|u^n\|_\infty = \max_{0 \le j \le M-1} |u_j^n|.$$

Before proposing the fully discrete numerical schemes, we have to show the following preparations.

Noting that

$$u^p \frac{\partial u(x,t)}{\partial x} = \frac{1}{p+2}\left[u^p(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial}{\partial x}u^{p+1}(x,t)\right],$$

we approximate the nonlinear advection term as follows

$$u^p \frac{\partial u}{\partial x}\bigg|_{(x,t)=(x_j,t_n)} = \frac{1}{p+2}\left[(u_j^{n-1})^p (u_j^n)_{\hat{x}} + ((u_j^{n-1})^p u_j^n)_{\hat{x}}\right]. \tag{2.1}$$

The way to approximate the nonlinear advection term is quite technical for two reasons. On one hand, thanks to the (2.1), the boundedness, stability and convergence of the numerical solutions are derived in the next section. In fact, if the advection term is approximated in the usual way, e.g., $(u_j^n)^p (u_j^n)_{\hat{x}}$, $(u_j^n)^p (u_j^n)_x$ and $(u_j^n)^p (u_j^n)_{\bar{x}}$, the mentioned numerical results can not be obtained easily. On the other hand, the numerical scheme is linear if (2.1) is applied. This point can be found clearly after the fully discrete numerical scheme is presented.

Next, for a better understanding of the fully discrete numerical schemes, we introduce the following lemma, which was first used to study the numerical solution of the diffusion-wave equation by Sun and Wu in [26].