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Numerical evaluation of the compact acoustic Green's function for scattering problems

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ABSTRACT

The reduction of noise generated by new and existing engineering products is becoming of increasing commercial importance. Noise prediction schemes are important tools available to help us understand and develop a means of controlling noise. Hybrid noise prediction schemes alleviate many issues associated with exclusively numerical or analytical approaches. These schemes often make use of a Green's function to compute the sound field – the Green's function representing geometrical scattering effects. Current hybrid schemes are limited to propagating noise in simple geometries for which the Green's function is known. In order to extend hybrid schemes to more general geometries, we develop here a robust, semi-analytical computational method to compute Green's functions for more general geometries in both 2D and 3D. The class of Green's functions considered here can be constructed through conformal mapping of the geometry to a canonical domain. Traditionally, this would only be possible if the mapping could be expressed analytically. Here we combine the traditional algorithm with a numerical mapping procedure to allow the Green's function to be computed for more general geometries. The accuracy is assessed through application to 2D benchmark problems for which analytical solutions are known. Although we assess the accuracy and speed of the method on 2D problems only, the extension to 3D only requires an additional execution of the same computational procedure for the extra dimension with a predictable effect on these two properties. We compute a Green's function for a baffle in a 2D channel, an important geometry in vortex sound problems, and a 3D projection from the half-plane. The semi- analytical method presented here demonstrates calculation of the Green's function accurately and robustly by avoiding particular conformal transformations and the evaluation of potential models containing singularities.

1. Introduction

Awareness of the effects of new and existing engineering products on the environment and on human health is of increasing commercial importance. The control of noise emissions is one of the ways the engineering industry can reduce the environmental impact of their products. A critical tool which enables understanding of noise emissions is that of noise prediction schemes. However, a robust noise prediction scheme, whether analytical or numerical, still remains far off. Part of the reason for this is the conflicting demands of real acoustic problems on analytical and numerical computational methods alike [1]. Acoustic problems usually feature a region of space containing a noise source which is often non-linear and difficult to represent analytically. The

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complex local sound field generated by the source is then propagated to an observer location some distance from the source region. This propagation requires numerical codes to be able to propagate the acoustic fluctuations over large distances despite their very low amplitude. Hence we require the combination of very high order numerical schemes over tight meshes to avoid numerical dissipation and maintain dispersion characteristics of the acoustic waves. In contrast, analytical techniques, the original tools for aero-acoustic analysis, excel at rapid and accurate propagation of known sources to an observer location without any of the difficulties encountered when using numerical schemes. However, their limitations include the ability to accurately model source behaviour and capture scattering effects due to boundaries of complex geometries.

Despite these issues, the inclusion of analytical techniques in *hybrid* prediction schemes has shown promise, with such schemes combining the advantages of numerical schemes in predicting non-linear sources with analytical propagation [2]. However, for this approach to be robust, we require a robust way to deal with the any scattering by physical boundaries in the propagation region between the source and observer. In this paper we explore a generalised method capable of numerically evaluating a Green's function which may be included in existing hybrid schemes to take account of the scattering effects of the physical boundaries.

The method presented here may be used to calculate the Green's function for general 2D geometries under a number of assumptions which leads to the approximation known as the compact Green's function [3]. A number of practical problems meet these assumptions [4] making calculation of the compact approximation to the exact Green's function for a problem an important pursuit. The inclusion of such a numerical Green's function in an existing hybrid noise prediction scheme will greatly increase their utility. Although we focus on 2D problems here for simplicity and validation purposes, more general 3D fields may easily be constructed by application of the technique to the extra dimensions where necessary, as we demonstrate later.

This paper extends the work presented by Harwood and Dupère [5]. First we outline the approach to approximate the Green's function under the conditions used by Howe [3]. We then develop the algorithm to perform the calculation of this approximate Green's function and apply it to the 2D test geometries used by Harwood and Dupère, whose cases have known analytical solutions for comparison. We then proceed to develop this earlier work further by providing quantification and illustration of the suggested improvements to the computational configuration. In addition, we compare the performance of the semi-analytical method to the direct collocation Boundary Element Method (BEM) illustrating its speed and accuracy. Finally, we describe the extension of the method to a 3D problem and provide a more realistic application by computing the compact Green's function for a baffle of two different thicknesses within a channel.

2. Method development

Acoustic problems may be modelled using an appropriate representation theorem such as one of the many acoustic analogies. Acoustic analogies are designed to express the governing equations of the problem in a form which separates the sources of the sound from the propagation mechanism. Lighthill [6] was the first to exactly rearrange the Navier–Stokes equations into an inhomogeneous wave equation where one side of the equation represented linear wave propagation of the acoustic density fluctuation and the other side a number of physical source mechanisms. For time-harmonic acoustic problems, the generation and propagation of acoustic waves in a fluid may be represented by the Helmholtz equation. To facilitate noise prediction in rigid geometries we can look to solving the Helmholtz equation governing an acoustic variable ϕ which fluctuates due to source q within an arbitrary geometry, subject to rigid wall conditions on the boundary.

$$(\nabla^2 + k^2)\hat{\phi}(\mathbf{x}, \omega) = \hat{q}(\mathbf{x}, \omega) \quad \frac{\mathrm{d}\phi}{\mathrm{d}n} = 0 \tag{1}$$

Eq. (1) may be solved by transforming it into an integral equation. If the source is considered to be a distribution of points over a source region with the position of the points given by $\mathbf{y}$, our integral equation is defined:

$$\hat{\phi}(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} \hat{G}(\mathbf{x}, \mathbf{y}, \omega)\hat{q}(\mathbf{y}, \omega)\mathrm{d}^3\mathbf{y} \tag{2}$$

The function $\hat{G}(\mathbf{x}, \mathbf{y}, \omega)$ is a fundamental solution of the singular form of the governing differential equation (Eq. (3)) which also satisfies all boundary conditions and is an exact Green's function. Physically the Green's function may be interpreted as an outgoing wave produced by an impulsive unit point source at $\mathbf{y}$, observed at $\mathbf{x}$.

$$(\nabla^2 + k^2)\hat{G}(\mathbf{x}, \mathbf{y}, \omega) = \delta(\mathbf{x} - \mathbf{y}) \tag{3}$$

In the absence of physical boundaries we may consider this wave to propagate in free-space with solution in 3D

$$\hat{G}(\mathbf{x}, \mathbf{y}, \omega) = \frac{-e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi\,|\mathbf{x} - \mathbf{y}|} \tag{4}$$

However, as we introduce walls and bodies into the domain, the Green's function should either satisfy the boundary conditions on these additional surfaces or additional terms are required. It should be noted at this stage that although we limit our treatment to rigid boundaries, porous walls may be incorporated into the Green's functions for low Mach number flows through the matching of different Green's functions at the porous boundary [7].