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Shifted fractional-order Jacobi orthogonal functions: Application to a system of fractional differential equations

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ABSTRACT

In this study, we propose shifted fractional-order Jacobi orthogonal functions (SFJFs) based on the definition of the classical Jacobi polynomials. We derive a new formula that explicitly expresses any Caputo fractional-order derivatives of SFJFs in terms of the SFJFs themselves. We also propose a shifted fractional-order Jacobi tau technique based on the derived fractional-order derivative formula of SFJFs for solving Caputo type fractional differential equations (FDEs) of order ν $(0 < \nu < 1)$. A shifted fractional-order Jacobi pseudo-spectral approximation is investigated for solving the nonlinear initial value problem of fractional order ν. An extension of the fractional-order Jacobi pseudo-spectral method is given to solve systems of FDEs. We describe the advantages of using the spectral schemes based on SFJFs and we compare them with other methods. Several numerical example are implemented for FDEs and systems of FDEs including linear and nonlinear terms. We demonstrate the high accuracy and efficiency of the proposed techniques.

1. Introduction

In the last few decades, the applications of fractional calculus have been extended to various areas such as control engineering [1,2], signal processing [3], biosciences [4], electromagnetism [5], fluid mechanics [6], diffusion processes [7], electrochemistry [8], continuum and statistical mechanics [9], the dynamics of viscoelastic materials [10], propagation of spherical flames [11], and pharmacokinetics [12]. Many researchers have considered the theory of the existence and uniqueness of FDEs, such as [13–16].

Developing analytical and numerical methods for the solutions of FDEs is a very important task. Indeed, it is difficult to obtain exact solutions for most FDEs. Therefore, attempts have been made to propose analytical methods that approximate the exact solutions of these equations, such as the Adomian decomposition [17], variational iteration [18], and homotopy perturbation [19] methods. Recently, numerical methods have also been proposed for solving FDEs [20–26].

In terms of accuracy, one of the best methods for obtaining the numerical solutions of various types of differential equations is the spectral method (see [27–31]). All types of spectral methods are global, so they are highly convenient for approximating the solutions of linear and nonlinear FDEs [32–35]. Doha et al. [36] presented and developed spectral tau and collocation techniques for solving multi-term FDEs including linear and nonlinear terms using Jacobi polynomials, where they generalized the quadrature Legendre tau method [37] and the Chebyshev spectral methods [38]. Recently, [39] proposed a new fractional-order Legendre orthogonal function based on Legendre polynomials for obtaining highly accurate solutions of FDEs in a finite interval.

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Yin et al. [40] proposed a two-dimensional version of fractional-order Legendre orthogonal functions and derived the operational matrices of derivative and integrals for these functions to solve the two-dimensional FDEs. In addition, to solve FDEs on a semi-infinite interval, Bhrawy et al. [41] derived the fractional integrals of a generalized Laguerre operational matrix to obtain approximate solution of the linear FDEs. Moreover, a system of FDEs with uncertainty was solved by the eigenvalue–eigenvector method in [42].

In this study, we define new orthogonal functions called shifted fractional-order Jacobi orthogonal functions (SFJFs) based on the shifted Jacobi polynomials, and we derive a new formula that explicitly expresses any Caputo fractional-order derivative of SFJFs in terms of the SFJFs themselves. Thus, we propose a direct technique for solving linear fractional differential equations (FDEs) of fractional order ν $(0 < \nu < 1)$ using the shifted fractional-order Jacobi tau method (SFJTM). We also propose a new shifted fractional-order Jacobi collocation method (SFJCM) for solving the fractional initial value problem of fractional order ν $(0 < \nu < 1)$ with nonlinear terms, where the nonlinear FDE is collocated at the N zeros of the SFJF defined on the interval [0, 1]. The resulting algebraic equations and one algebraic equation obtained from treating the initial condition comprise $(N+1)$ nonlinear algebraic equations, which can then be solved by implementing Newton's iterative technique to find the unknown SFJFs coefficients. We extend the application of SFJCM based on SFJFs to solve a system of linear FDEs with fractional orders of less than 1. The spectral approximations based on fractional-order Chebyshev functions and fractional-order Legendre functions [39] are special cases of the proposed approximations, and the proposed approach contains many other special cases. Several numerical examples are implemented to confirm the high accuracy and effectiveness of the proposed method for solving FDEs of fractional order ν $(0 < \nu < 1)$. Our methods extend and improve existing methods that have been reported previously.

The remainder of this paper is organized as follows. First, we present some necessary definitions of the fractional calculus in Section 2. In Section 3, we define the SFJFs. In Section 4, we derive the main theorem of this study, which provides a new formula that explicitly expresses the fractional-order derivative of the SFJFs in terms of the SFJFs themselves. In Section 5, we apply spectral methods based on SFJFs to solve FDEs and systems of FDEs including linear and nonlinear terms of fractional order less than 1. Several examples are presented in Section 6 to illustrate the main ideas in this study based on comparisons. We give our conclusions in Section 7.

2. Preliminaries and notations

In this section, we give some basic definitions and properties of fractional calculus theory, which are used in this study.

Definition 2.1. The Riemann–Liouville fractional integral operator of order $\nu \geq 0$ is given by

$$\begin{aligned} J^{\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \quad \nu > 0, \quad x > 0, \\ J^0 f(x) &= f(x). \end{aligned} \tag{2.1}$$

Definition 2.2. The Caputo fractional derivative of order ν is defined by

$$\begin{aligned} D^{\nu} f(x) = J^{m-\nu} D^m f(x) = &\frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} \frac{d^m}{dt^m} f(t)dt, \\ &m-1 < \nu \leq m, \; x > 0, \end{aligned} \tag{2.2}$$

where m is the smallest integer greater than ν.

The operator D^{ν} satisfies the following properties

$$D^{\nu} C = 0, \quad (C \text{ is a constant}), \tag{2.3}$$

$$D^{\nu} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}, & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil \nu \rceil \text{ or } \beta \notin N \text{ and } \beta > \lfloor \nu \rfloor, \end{cases}$$

where $\lceil \nu \rceil$ and $\lfloor \nu \rfloor$ are the ceiling and floor functions, respectively, while $N = \{1, 2, \ldots\}$ and $N_0 = \{0, 1, 2, \ldots\}$.

Caputo's fractional differentiation is a linear operation,

$$D^{\nu}(\lambda f(x) + \mu g(x)) = \lambda D^{\nu} f(x) + \mu D^{\nu} g(x), \tag{2.5}$$

where λ and μ are constants.

3. Fractional-order Jacobi functions

In this section, we present some useful properties of Jacobi polynomials. Next, we define the SFJFs.