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Smoothing approach for a class of nonsmooth optimal control problems

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ABSTRACT

In this study, we consider a nonsmooth optimal control problem. First, we convert this problem into the corresponding smooth optimal control problem using a practical generalized derivative. Next, we utilize the Chebyshev pseudo-spectral method to solve the smooth problem and analyze the feasibility and convergence of the approximations obtained. Finally, we approximate the optimal solutions of some nonsmooth optimal control problems.

1. Introduction

Optimal control (OC) problems, including nonlinear ordinary differential equations (ODEs), occur in many applications such as mechanics, economics, robotics, and aeronautics. There are two general classes of methods for solving OC problems: direct methods and indirect methods. Direct methods are based on discretization and parameterization, which lead to nonlinear programming. The class of direct methods includes quasilinearization methods, steepest descent methods, quasi-Newton approximations methods, spectral and pseudo-spectral methods, algorithmic differentiation methods, finite difference methods, measure theoretical approaches, linearization methods, control parameterization methods, and time-scaling transformation methods, (e.g., see [1–15]). However, indirect methods are based on the Pontryagin minimum (or maximum) principle and Hamiltonian-Jacobi-Bellman equations, which can lead to problems with initial or boundary values due to the necessary optimality conditions for OC. We can solve the initial or boundary problems in these methods by using collocation methods [1,16] and spectral and pseudo-spectral methods [17–20]. However, some OC problems involve a nonsmooth dynamical system. The nonsmooth dynamical systems (see [21–34]) include discontinuous functions or continuous but nondifferentiable functions. Some well-known nonsmooth dynamical systems are nonsmooth electrical circuits, mechanical systems with Coulomb friction and impact, ODEs with discontinuous right-hand side, and switching systems. Queiroz et al. [35] presented a collection of research results dealing with nonsmoothness in OC problems. In nonsmooth OC problems, we cannot utilize the direct and indirect methods described above because these methods usually require the differentiation, gradient, Hessian, and Jacobian of functions to obtain the optimal solution.

The major developments in nonsmooth OC problems are related to the necessary conditions for optimality based on nonsmooth analysis (see [36,37]). The techniques employed to derive these conditions involve expansions of the studies by Rockafellar

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[37–41], Clarke [42–45], Ioff [46–48], Loewen and Rockafellar, [49,50], Mordokovich [51,52], and Vinter [53]. Moreover, in these developments, the nonsmooth OC problems are usually converted into OC problems that involve inclusion differential equations. However, we cannot use these optimality conditions to solve the nonsmooth OC problem in practical and numerical terms. In fact, these conditions are employed to test the optimality of a given solution. For instance, the necessary conditions for optimality were presented by Vinter (see pages 203, 206, and 214 in [53]). In addition, there are some numerical methods for solving nonsmooth OC problems that involve a special dynamical system (see [54–56]).

Hence, in the present study, we propose an applied direct approach for obtaining an approximate optimal solution for a class of nonsmooth OC problems, as follows:

$$\text{Minimize} \quad J(x(.), u(.)) = \rho(x(b)) + \int_a^b f(t, x(t), u(t))\, dt \tag{1}$$

$$\text{subject to} \quad \dot{x}(t) = h(t, x(t)) + p(t, x(t), u(t)), \; t \in [a, b], \tag{2}$$

$$x(a) = \alpha, \quad x(t) \in A, \quad u(t) \in B, \quad t \in [a, b] \tag{3}$$

where a, b are the given real numbers, $\alpha \in \mathbb{R}^n, x(.) : [a, b] \to A \subseteq \mathbb{R}^n$ is the state variable, and $u(.) : [a, b] \to B \subseteq \mathbb{R}^m$ is the control variable. Moreover, we consider the following assumptions for the nonsmooth OC problem (1)–(3).

I. Functions, $\rho(.), f(., ., .)$ and $p(., ., .)$ are differentiable (or smooth) with respect to their arguments.
II. $h(., .)$ is a continuous nonsmooth function and it is not constant with respect to its second argument. Moreover, the set of nonsmoothness points of $h(t, .)$ for all $t \in [a, b]$ and $h(., x)$ for all $x \in A$ is countable. Furthermore, we assume that $h(., .)$ is smooth in point (a, α).
III. The ODE (2) holds Lebesgue almost everywhere (a.e.) (see Folland [57]) on the interval $[a, b]$.

We say that a state-control pair $(x(.), u(.))$ is admissible if the following conditions hold.

(1) The state $x(.)$ satisfies $x(t) \in A, \; t \in [a, b]$ and is differentiable on $[a, b]$.
(2) The control $u(.)$ satisfies $u(t) \in B, \; t \in [a, b]$ and is piecewise continuous on $[a, b]$.
(3) The condition $x(a) = \alpha$ is satisfied.
(4) The pair $(x(.), u(.))$ satisfies the differential Eq. (2) Lebesgue a.e. on the interval $[a, b]$ in the sense of Caratheodory.

Furthermore, we can say that the admissible pair $(x^*(.), u^*(.))$ is an optimal solution to the nonsmooth OC problem (1)–(3) when $J(x^*(.), u^*(.)) \le J(x(.), u(.))$ for any admissible pair $(x(.), u(.))$.

In this study, we use the practical generalized derivatives (GDs) of nonsmooth functions, which were proposed in [58–60] and we describe them briefly. We then employ these generalized derivatives to convert the nonsmooth OC problem (1)–(3) into the corresponding form. We utilize the Chebyshev pseudo-spectral (CPS) method and we approximate the obtained smooth OC problem by the finite dimensional nonlinear programming (NLP) problem. By solving the latter NLP problem, we can approximate the optimal state and the OC of the nonsmooth OC problem (1)–(3).

The remainder of this paper is organized as follows. In Section 2, we introduce a practical generalized derivative for nonsmooth functions. In Section 3, we convert the nonsmooth OC problem (1)–(3) into the corresponding smooth OC problem. In Section 4, we approximate the smooth OC problem by the NLP problem using the CPS method. In Section 5, we analyze the feasibility and convergence of the problem obtained. In Section 6, we approximately solve some nonsmooth OC problems using our proposed approach. In Section 7, we give our conclusions.

2. A practical generalized derivative

There are many GDs and subdifferentials for a special class of nonsmooth functions, which were introduced by Rockafellar [37–41], Clarke [42–45], Ioff [46–48], Mordukhovich [51,52], and others. These GDs are examples of the Jeyakumar pseudo-Jacobians (see [61]) and we generally cannot use these GDs to solve continuous and discrete-time nonsmooth optimization problems in a numerical manner. Of course, other practical GDs were described in [62,63], which can be used to solve the nonsmooth ODEs. Here, we briefly introduce a practical GD proposed by Noori Skandari et al. [58–60], which utilize in the following section.

Let Ω be a connected and compact set and $L : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a continuous nonsmooth function. Assume that $C(\Omega)$ and $C^1(\Omega)$ are the spaces of continuous and continuous differentiable functions on the set Ω, respectively. We assume that $\varphi_j(.), j = 0, 1, 2, \ldots,$ are the continuously differentiable basic functions for the space $C(\Omega)$ and we suppose that $N_\delta(s)$ is the neighborhood of s with radius δ. Divide the set Ω into similar sets $\Omega_i, \; i = 1, 2, \ldots, m$ (where m is a sufficiently big number) such that $\Omega = \cup_{i=1}^m \Omega_i$ and $\text{int}(\Omega_i) \cap \text{int}(\Omega_j) = \phi$ for $i \neq j$ (the notation int denotes the interior of the set). In addition, select the arbitrary points $s_i \in \text{int}(\Omega_i), \; i = 1, 2, \ldots, m$. Now, consider the following optimization problem:

$$\underset{a_{jk} \in \mathbb{R}}{\text{Minimize}} \quad \sum_{i=1}^m \int_{N_\delta(s_i)} \left| L(x) - L(s_i) - \sum_{k=1}^n \sum_{j=0}^\infty (x_k - s_{ik}) a_{jk} \varphi_j(s_i) \right| dx, \tag{4}$$