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Operation matrix method based on Bernstein polynomials for the Riccati differential equation and Volterra population model

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ABSTRACT

In this study, we present a modified configuration, including an exact formulation, for the operational matrix form of the integration, differentiation, and product operators applied in the Galerkin method. Previously, many studies have investigated the methods for obtaining operational matrices (derivative, integral, and product) for Fourier, Chebyshev, Legendre, and Jacobi polynomials, and some have considered the non-orthogonal bases that almost all of them operate on approximately. However, in this study, we aim to obtain the exact operational matrices (EOMs), which can be used for many classes of orthogonal and non-orthogonal polynomials. Similar to previous approaches, this method transforms the original problem into a system of nonlinear algebraic equations. To retain the simplicity of the procedure, the samples are considered in one-dimensional contexts, although the proposed technique can also be employed for two- and three-dimensional problems. Two examples are presented to verify the accuracy of the proposed new approach and to demonstrate the superior performance of EOMs compared with ordinary operational matrices. The corresponding results demonstrate the increased accuracy of the new method. In addition, the convergence of the EOM method is studied numerically and analytically to prove the efficiency of the method.

1. Introduction

1.1. Conventional methods

Orthogonal functions and polynomials have been employed widely for solving various problems, such as ordinary differential equations (ODEs) [1–7], partial differential equations (PDEs) [8–13], integral equations [14–17] and other types of equations [18–20]. The main aim of this approach is to reduce the problem to solving a system of algebraic equations, which is obviously a great simplification. This approach is based on the approximation of all of the functions used in the system as a truncated polynomial series (e.g., $F_m(x) = \{f_0(x), f_1(x), f_2(x), \ldots, f_m(x)\}$, where $F_m(x)$ is called the basis set and $f_i(x)$s are called basis polynomials, such as Chebyshev or Legendre polynomials). Operational matrices (integral, product, and differentiation matrices) are then used to omit the corresponding operators.

The main method for implementing this approach is to first approximate all of the known functions as well as the unknown function (e.g., $y(x)$) to ensure that they are located within the $span(F_m(x))$, before performing operations on $y(x)$ (such as integration and product) using ordinary operational matrices (OOMs).

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To illustrate the concept of OOMs, we first define a vector $\phi_m(x)$ comprising elements of $F_m(x)$

$$\phi_m(x) = [f_0(x)\ f_1(x)\ f_2(x)\ \cdots\ f_m(x)]^T. \tag{1}$$

The integral operational matrix for the basis vector $\phi_m(x)$ operates as

$$\int_0^x \phi_m(t)dt \simeq P_{m\times m}\phi_m(x), \tag{2}$$

and for the product operational matrix as,

$$\phi_m(x)\phi_m(x)^T c \simeq \widehat{C}\phi_m(x), \tag{3}$$

as well as for the differentiation operational matrix

$$\frac{d}{dx}\phi_m(x) = D_{m\times m}\phi_m(x). \tag{4}$$

The main function of OOMs is to conduct the desired operations so the residual remains in $span(F_m(x))$.

Each time a term is approximated, the accuracy will quite probably decrease. In other words, if the term (function) is outside of $span(F_m(x))$, then using its projection on $span(F_m(x))$ decreases the accuracy of the residual.

Therefore, using this method for solving a dynamical system may lead to repeated approximations before reaching the residual function of the system. For example, in relation (1), let $f_i(x) = x^i$ and $y(x) = c^T\phi_{10}(x)$. If we want to approximate $y^3(x)$ using OOMs, we have

$$y^3(x) \simeq c^T\left(\hat{C}\right)^2\phi_{10}(x) \in span\{F_{10}(x)\},$$

but we know that $y^3(x) \in span\{F_{30}(x)\}$. This means that there is a huge loss of dimension in the approximation.

1.2. A new idea

In our alternative approach, we do not need to perform these repeated approximations. The only function that is approximated must be located in $span(F_m(x))$ is $y(x)(\simeq c^T\phi_m(x),\ c = \begin{bmatrix} c_0 & c_1 & \cdots & c_m \end{bmatrix}^T)$, which is because of the nature of the approach.

In this study, we propose a new method for implementing this approach with far fewer restrictions. In this new method, although all of the terms must be approximated in $span(F_n(x))$, we can freely choose the arbitrary n (as large as we need) for each term.

Therefore, during the integration process, we do not restrict the result to being located in the $span(F_m(x))$ and it can also be in the $span(F_n(x))$, $(n > m)$ for an arbitrary n. Thus, we can have an exact integration matrix

$$\int_0^x \phi_m(t)dt = P_{m\times m+1}\phi_{m+1}(x). \tag{5}$$

Moreover, to multiply two unknown functions, we can introduce the exact product matrix

$$\phi_p(x)\phi_q(x)^T c = \phi_{p+q}^T(x)\widehat{C_{(p+q)\times q}}. \tag{6}$$

We can even reduce the dimension of the function space. Hence, we can have the following differentiation matrix

$$\frac{d}{dx}\phi_m(x) = D_{m\times m-1}\phi_{m-1}(x), \tag{7}$$

and without loss of generality, we suppose that the system has just one equation (if there are more, then the procedure is also applied to them). By applying the new method, each term of the equation is approximated by a basis vector (e.g., $d_n^T\phi_n(x)$, where d is a vector of length n that approximates one of the terms of the system). To obtain the residual function of the dynamical system, we must sum all of the approximated terms. To allow us to sum all of the different sized matrices, all of the terms should be the same size, such as $d'^T_N\phi_N(x)$ (N is the maximum n among all). This is achieved using "the increaser matrix," which we introduce in Section 2.4.2.

By summing all of the terms, we obtain the $Residual(x) = \sum_{i=0}^{nt} d'_N\psi_N(x) = R_{1\times(N+1)}\phi_N(x)$. Thus, $R_{1\times(N+1)}$ will be a vector function on c_is (elements of the unknown vector c), as introduced at the beginning of this section. To solve the system, we must set c_is so the best residual will be obtained. In this study, our interpretation of the "best residual function" or "the nearest residual function to zero" for a specific norm $\|\cdot\|$ is the function $r(x)$ such that

$$\forall\, f(x) \in span(F_m(x)) :\quad \|r(x)\| \le \|f(x) - r(x)\|. \tag{8}$$

Thus, $r(x)$ is a function where its best approximation in $span(F_m(x))$ is the zero function. This is obtained by solving the algebraic system

$$R_m^* = R_{1\times(N+1)}Q_{N,m} = 0, \tag{9}$$

where the matrix $Q_{N,m}$ (named Q matrix) is described in Section 2.4.7. By solving the algebraic system above, we find the m unknown coefficients (elements of vector c_m), and thus we obtain the final solution ($y(x) \simeq c_m^T \times \psi_m(x)$).