Applied Mathematical Modelling 40 (2016) 1097–1109

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Two-grid variational multiscale method with bubble stabilization for convection diffusion equation

Zhifeng Weng [a], Jerry Zhijian Yang [a,b], Xiliang Lu [a,b,*]

[a] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, PR China
[b] Computational Science Hubei Key Laboratory, Wuhan University, Wuhan, 430072, PR China

ARTICLE INFO

Article history:
Received 20 March 2014
Revised 4 May 2015
Accepted 17 June 2015
Available online 21 July 2015

Keywords:
Convection diffusion equation
Variational multiscale method
Two-grid method
Local Gauss integrations
Bubble functions

ABSTRACT

A two-grid variational multiscale (VMS) method based on two local Gauss integrations for the convection dominated nonlinear convection diffusion equation is investigated. This method combines the two-grid strategy with the variational multiscale method which chooses polynomial bubble functions as subgrid scale. Two local Gauss integrations are applied to replace the projection operator without adding any extra storage. Moreover, the error estimates for the algorithms of the two-grid method are obtained. Numerical examples validate the theoretical results of the presented methods.

1. Introduction

The convection diffusion equations describe a lot of physical phenomenons in computational fluid dynamics such as chemical reaction processes, heat conduction, nuclear reactors, population dynamics, just to name a few. The governing equations are often with a nonlinear source or sink term. In this work we consider the following stationary nonlinear convection diffusion equations:

$$\alpha \cdot \nabla u - \varepsilon \Delta u = f(u), \quad \text{in } \Omega, \tag{1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary $\partial\Omega$. The constant ε is the diffusion coefficient and α is either a constant vector or a divergence free velocity field. The nonlinear reaction terms $f(u)$ appear usually in the form of products and rational functions of concentrations, or exponential functions of the temperature, expressed by the Arrhenius law in chemical engineering.

When $\varepsilon \ll 1$, Eq. (1)–(2) is a convection dominated problem. The standard continuous piecewise linear finite element method may produce a large non-physical oscillations to the approximate solutions, unless the mesh size is small enough with respect to the diffusion coefficient. In order to overcome the numerical instabilities, people introduce different stabilization methods, such as, the Galerkin least square (GLS) method [1,2], stream upwind Petrov Galerkin (SUPG) method [3,4], the residual-free bubbles (RFB) method [5,6], the local projection stabilization [7,8] and etc. for linear convection dominated problems. For nonlinear convection dominated problems, Markus Bause in [9] provided SUPG stabilized higher-order finite element approximations of

* Corresponding author.
E-mail addresses: zfweng@whu.edu.cn (Z. Weng), zjyang.math@whu.edu.cn (J.Z. Yang), xllv.math@whu.edu.cn (X. Lu).

http://dx.doi.org/10.1016/j.apm.2015.06.023

convection diffusion reactions models with nonlinear reaction mechanisms. Markus et al. [10,11] analyzed the numerical performance properties of higher order finite element approaches along with SUPG and additional shock-capturing stabilization for nonlinear convection dominated problem. Yücel et al. [12] studied convection diffusion reaction models with nonlinear reaction mechanisms by using the discontinuous Galerkin method and the upwind symmetric interior penalty Galerkin (SIPG) method.

However, the SUPG methods have some drawbacks in application: it brings in additional nonphysical coupling term and hence unphysical oscillations near the boundary; and the second order derivative is needed for the high order finite element approximations. To overcome those drawbacks of SUPG methods, alternative some stabilization methods have been reformulated in the framework of the variational multiscale method. Hughes [13] proposed the variational multiscale method (VMS) to overcome spurious oscillations in solutions which is caused by the multiscale structure of the problem. The VMS method decomposes the solution into large scale and small scale such that one can obtain a coupled system of two sub-problems for the different scales. In such way one can obtain a stabilized formulation. There are different ways to define the large scale problem in the variational multiscale method, for example, projection into appropriate subspaces is applied in Hughes et al. [14,15], Guermond [16], Layton [17,18] and Zheng et al. [19]. Especially, Song et al. [20] proposed a variational multiscale method based on polynomial bubble functions as subgrid scale for linear convection dominated convection diffusion equation. Similar as the work of Song et al. [20], we extend this method to solve the nonlinear convection diffusion problem. More precisely, we add an eddy viscosity stabilization which chooses a bubble function as subgrid scale, and the projection operator is reformulated by the local Gaussian quadrature. It keeps the same efficiency without adding any extra storage compared with common VMS in [21].

To increase the efficiency of a numerical method, an alternative idea is the two-grid method. The two-grid discretization strategy is to compute the nonlinear equation on a coarse mesh, then to solve a linearized system (at the solution from coarse mesh) on a fine mesh. The two-grid method can be found in the works of Xu [22,23], Chen et al. [24], He et al. [25–27], Zhang and He [28], Shang [29], Huang et al. [30], Weng et al. [31] and etc.

In this paper we combine the two-grid method with the variational multiscale method for solving the two-dimensional steady convection diffusion problem based on the locally stabilized method with Gaussian quadrature rule. The remainder of this paper is organized as follows. In Section 2, a variational multiscale finite element method is introduced. The two-grid method is given in Sections 3. In Section 4, numerical experiments are given to validate the theoretical results.

The standard Sobolev space $W^{m,p}(\Omega)$ is equipped with a norm $\|\cdot\|_{m,p}$. For $p=2$, let $H^m(\Omega)=W^{m,2}(\Omega)$ and write $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$. We use the constant C or c to denote a generic positive constant whose value may change from place to place, but remains independent of the mesh parameter and ε.

2. Variational multiscale finite element method

2.1. Variational formulation of the convection–diffusion problem

Define bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $B(\cdot,\cdot)$ on $H_0^1(\Omega)\times H_0^1(\Omega)$ by

$$a(u,v)=\varepsilon(\nabla u,\nabla v),\ b(u,v)=(\alpha\cdot\nabla u,v),\ B(u,v)=a(u,v)+b(u,v).$$

Then the variational formulation of problem (1)–(2) reads: find $u\in H_0^1(\Omega)$ such that

$$B(u,v)=(f(u),v),\quad \forall v\in H_0^1(\Omega). \tag{3}$$

We assume the nonlinear term f is a locally Lipschitz, monotone function and is bounded up to second order derivative, i.e. for any $t,t_1,t_2\ge 0, t,t_1,t_2\in\mathbb{R}$ the following conditions

$$\|f(t_1)-f(t_2)\|\le L\|t_1-t_2\|,\quad L>0 \tag{4}$$
$$f\in C^2(\mathbb{R}),\quad f'(t)\le 0,\quad |f(t)|+|f''(t)|\le C$$

So that (3) admits a unique solution$u\in H_0^1(\Omega)$ (c.f. [16]).

2.2. Variational multiscale finite element method based on two local Gauss integrations

Let $M_h\subseteq H_0^1(\Omega)$ be a standard piecewise polynomial finite element defined on Ω and let u_h be the finite element approximation solution of problem (3), and satisfy

$$B(u_h,v_h)=(f(u_h),v_h)\quad \forall\ v_h\in M_h. \tag{5}$$

It is known that the formulation (5) lacks coercivity when $\varepsilon << |\alpha|$, unless the mesh size h is small enough with respect to the diffusion coefficient ε. The common variational multiscale methods in [21] to stabilize (5) is as follows: let $\mathbf{L}_{\tilde h}$ be a vector valued finite element subspace of $[L^2(\Omega)]^2$, to find $u_h\in M_h$, $\mathbf{P}_{\tilde h}\in\mathbf{L}_{\tilde h}$ satisfying

$$\begin{aligned} B(u_h,v_h)+(\nu\nabla u_h,\nabla v_h)-(\nu\mathbf{P}_{\tilde h},\nabla v_h)=(f(u_h),v_h),\quad \forall v_h\in M_h,\\ (\mathbf{P}_{\tilde h}-\nabla u_h,\mathbf{I}_{\tilde h})=0,\quad \forall\mathbf{I}_{\tilde h}\in\mathbf{L}_{\tilde h}.\end{aligned} \tag{6}$$

The second equation in (6) implies that $\mathbf{P}_{\tilde h}$ is the L^2-orthogonal projection of ∇u_h into $\mathbf{L}_{\tilde h}$. The space $\mathbf{L}_{\tilde h}$ contains the information of the coarse scale, and the stabilization parameter ν acts only on the fine scale. The choice of ν is discussed in [21]. The constant