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On systems of nonlinear equations: some modified iteration formulas by the homotopy perturbation method with accelerated fourth- and fifth-order convergence

K. Sayevand [a,*], H. Jafari [b,c]

[a] Faculty of Mathematical Sciences, Malayer University, Malayer, Iran
[b] Department of Mathematics, Mazandaran University, Mazandaran, Iran
[c] Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA 0003, South Africa

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ABSTRACT

This paper is concerned with the development of efficient algorithms for solving the system of nonlinear equations based on the modified homotopy perturbation method. The analysis of convergence shows that the proposed methods are fourth and fifth-order convergence. Several numerical examples are presented. Comparisons are made to confirm the reliability and effectiveness of the proposed schemes.

1. Introduction

Systems of nonlinear equations often appear in numerical applications and are usually difficult to solve, either exactly or numerically [1–5]. In recent years, several iterative methods have been developed to solve the nonlinear system of equations. The most commonly used ones are; the Adomian decomposition method [6], quadrature formulas [7–9], the homotopy perturbation method (HPM) [10] and other techniques [11–18]. In this paper, we develop two iterative methods based on the modified homotopy perturbation method (MHPM) [19,20] to solve the system of nonlinear equations. Some related attempts at considering the behavior of extended homotopy perturbation method for nonlinear systems have been made by Sayevand and Fardi [2]. On the basis of these attempts, the aim of the present work is to extend the analysis of Ref. [2]. In this paper we prove that these new methods have fourth- and fifth-order convergence. Several numerical examples are given to illustrate the efficiency and the performance of the new iterative methods. The obtained results suggested that these new methods introduce a powerful improvement for solving nonlinear equations.

* Corresponding author.
E-mail addresses: ksayehvand@malayeru.ac.ir (K. Sayevand), jafari@umz.ac.ir, jafarh@unisa.ac.za (H. Jafari).

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2. Analysis of the iterative algorithms

Consider the following nonlinear system

$$\Phi(X)=\begin{cases} f_1(X)=0,\\ f_2(X)=0,\\ \vdots\\ f_N(X)=0, \quad X=(x_1,x_2,\dots,x_N)^T\in\Re^N. \end{cases} \tag{1}$$

Let $\overline{X}=(\overline{x_1},\overline{x_2},\dots,\overline{x_N})^T\in\Re^N$, be an estimation of a zero of this nonlinear system and

$$\Phi(X):\Re^N\to\Re^N, \tag{2}$$

be a sufficiently differentiable function (its rate of changes is characterized by conditions on derivatives: see ([1], pp. 298)). Using the Taylor series, we can write the nonlinear system as follows

$$\Phi(\overline{X})+\Phi'(\overline{X})(X-\overline{X})+R(X)=0, \tag{3}$$

where

$$R(X)=\Phi(X)-\Phi(\overline{X})-\Phi'(\overline{X})(X-\overline{X}), \tag{4}$$

and $\Phi'(X)=[\frac{\partial f_i(X)}{\partial x_j}]|_{i,j=1}^N$ is the Jacobian matrix of $\Phi(X)$. It is to be noted that these properties ([3,4]) were discussed in details in the monograph [3]. Now, without loss of generality assume that the inverse of Jacobian matrix of $\Phi'(X)$ at $\overline{X}$ exists and is bounded (the reader is advised to consult the results of the research works presented in [5]), i.e.

$$\|[\Phi'(\overline{X})]^{-1}\|\le\eta, \quad \eta>0, \tag{5}$$

where

$$[\Phi'(X)]^{-1}=[\Psi_{i,j}(X)]|_{i,j=1}^N. \tag{6}$$

In Eq. (1) for $N=1$, we have

$$\Phi(X)=f_1(X)=0. \tag{7}$$

As a direct consequence of (1) we will have

$$f_1(X)f_1^{-1}(\bar{X})=0, \tag{8}$$

consequently, one will set

$$X=\bar{X}-f_1(\bar{X})f_1^{-1}(\bar{X})+X-\big(\bar{X}-f_1^{-1}(\bar{X})f_1(\bar{X})\big)-f_1^{-1}(\bar{X})f_1(X), \tag{9}$$

In the other words, in general for $f_1=f_2=\cdots=f_N=0$ and by using (6) we obtain

$$0=-\sum_{n=1}^{N}\Psi_{i,n}(\bar{X})f_n(X), \quad i=1,2,\dots,N. \tag{10}$$

Whence (10) implies that

$$x_i=x_i-\sum_{n=1}^{N}\Psi_{i,n}(\overline{X})f_n(X)+\sum_{n=1}^{N}\Psi_{i,n}(\bar{X})f_n(\bar{X})-\sum_{n=1}^{N}\Psi_{i,n}(\bar{X})f_n(\bar{X}), \quad i=1,2,\dots,N. \tag{11}$$

It is easily verified that Eq. (11) can be written in the form of following equations

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_N\end{pmatrix}=\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_N\end{pmatrix}+\begin{pmatrix}N_1(X)\\N_2(X)\\\vdots\\N_N(X)\end{pmatrix}, \tag{12}$$

where

$$\alpha_i=\overline{x_i}-\sum_{n=1}^{N}\Psi_{i,n}(\overline{X})f_n(\overline{X}), \tag{13}$$

and

$$N_i(X)=x_i-\alpha_i-\sum_{n=1}^{N}\Psi_{i,n}(\overline{X})f_n(X). \tag{14}$$