Applied Mathematical Modelling 40 (2016) 1676–1682

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Short communication

On the construction of minimal surfaces from geodesics

J. Sánchez-Reyes*

IMACI, ETS Ingenieros Industriales, UCLM. Campus Universitario, 13071 Ciudad Real, Spain

ARTICLE INFO

Article history:
Received 26 January 2014
Revised 9 March 2015
Accepted 3 July 2015
Available online 4 August 2015

Keywords:
Björling's problem
Catenoid
Geodesic
Minimal surface
Plateau's problem
Helicatenoid

ABSTRACT

In a recent article (2013), Li et al. [1] approximate minimal surfaces from geodesic boundaries, with applications to garment design in mind. We go over this work and existing methods for constructing minimal surfaces from geodesics. First, we justify why minimal surfaces and the problem of finding the surface with minimal area (i.e., solving Plateau's problem) have little to do with garment design. Second, we recall that Plateau's problem makes little sense for boundaries such as those considered in [1], composed of unclosed curves of finite length or disconnected pieces of them (with no other positional restriction). Finally, we note that the construction of a minimal surface (with zero mean curvature) from a prescribed geodesic is a particular instance of a classical problem in differential geometry, already solved by Björling. In particular, for a geodesic circle or helix the resulting minimal surfaces are well-known (catenoid and helicatenoid, respectively), so no approximations are required.

1. Introduction

The constrained construction of parametric surfaces from one or more prescribed geodesics has received ample attention in recent literature. A key motivation behind this research is garment design, where a given model and size is characterized by a curve, called girth (Fig. 1), that the designer must preserve [2]. When the fabric or leather is flattened, it is convenient that the girth be a geodesic to minimize distortion [3]. Furthermore, shoemakers naturally employ geodesics to cut patterns [4]. In a seminal work, Wang et al. [2] show how to define a pencil of surfaces having a geodesic as isoparametric curve (i.e., an *isogeodesic*). This work has been extended to the case of a rational pencil [5], and polynomial surfaces from Bézier isogeodesics [6,7]. Another motivation for the research on surfaces from isogeodesics is the development of a certain ribbon with embedded micro-sensors [8–12]. When a ribbon is placed tight on a surface, it naturally follows a geodesic, as sketched in Fig. 2, since it tries to minimize the distance. Given a physical surface, the aforementioned ribbon with micro-sensors provides tangential information that can be employed for reconstruction purposes, furnishing an innovative alternative to traditional scanning methods, which only capture positional data. On the other hand, finding surfaces with minimal area under a boundary constraint is a classical problem in differential geometry [13,14]. Since minimal surfaces display remarkable properties, they find numerous applications in CAD, especially in architectural design [15] and aesthetic engineering [16].

In principle, the idea of combining minimal surfaces and geodesic boundaries may appear attractive. Thus, in a recent article Li et al. [1] tackle the approximation of minimal surfaces from geodesics, for applications in garment and shoe design, in order to minimize material consumption. We examine this construction and alternative methods for defining minimal surfaces from geodesic boundaries. First, a closer look at the problem (Section 2) reveals that the minimum area constraint makes little sense in the context of clothing design. In Section 3 we note that, for a boundary composed of disconnected unclosed curves of finite

* Tel.: +34 926 295463; fax: +34 926 295361.
E-mail address: Javier.SanchezReyes@uclm.es

http://dx.doi.org/10.1016/j.apm.2015.07.011

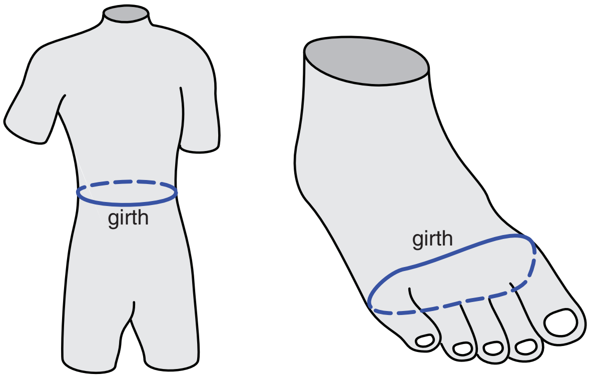

Fig. 1. Examples of girth curves for garment design.

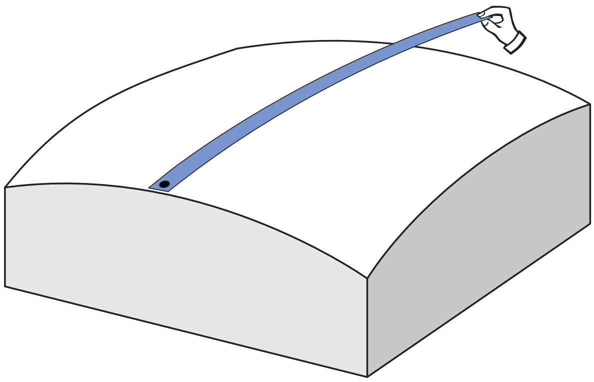

Fig. 2. Geodesic as the trajectory followed by a tight ribbon on a surface.

length, it has neither sense to find a surface with minimal area (as in several examples in [1]), because the lower bound of the area for surfaces having the prescribed boundary is precisely zero. Then, in Section 4, we analyze the geodesic condition along a boundary, and then recall (Section 5) that the construction of a minimal surface containing a given geodesic is a particular instance of the so-called Björling's problem. In particular, in Section 6 we observe that the minimal surfaces from a prescribed geodesic circle or a helix are well-known. Finally, conclusions are drawn in Section 7.

2. Minimal surfaces: physical interpretation and applications

Given a simple closed boundary Γ, finding the surface with minimal area is called *Plateau's problem* [17,18]. Such *minimal surfaces* correspond to the shape assumed by thin soap films spanning Γ, as long as the air pressure at both sides of the film is the same. Experimentally, minimal surfaces can be generated by twisting a wire following the contour Γ, then dipping it into a soap solution and carefully removing the wire. For an excellent and accessible introduction to the fascinating subject of minimal surfaces and their physical interpretation, the reader is addressed to the beautifully illustrated book by Hildebrandt and Tromba [19].

If we disregard the negligible weight of the film, according to Laplace's equation the pressure difference between the sides of a soap film is proportional to its mean curvature H (i.e., the arithmetic mean of the principal curvatures κ_1, κ_2). Therefore, minimal surfaces display necessarily a vanishing mean curvature everywhere:

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = 0. \quad (1)$$

Two important remarks regarding the condition above:

- It is necessary, but not sufficient, for a surface to *globally* minimize area. There exist surfaces that do not globally minimize the area in a Plateau's problem and yet satisfy (1), although in practice they would be unstable and hence not obtainable as soap films.
- Nevertheless, condition (1) is necessary and sufficient to *locally* minimize area, i.e., so that each point on the surface has a neighborhood with least-area relative to its boundary. Thus, it has become customary [19] to denominate *minimal* all surfaces with vanishing H, a convention that will be followed throughout this paper.

Minimal surfaces can be materialized with thin membranes, carrying only tension and no compression or bending. These tensile structures find remarkable applications in architecture, as tents and lightweight roofs, since they can span large distances