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Convergence of variational iteration method applied to two-point diffusion problems

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ABSTRACT

A new approach is introduced to construct the variational iteration algorithm-II for diffusion problems. Based on this algorithm, the sufficient conditions for convergence of variational iteration method applied to this type of two-point boundary value problems are established. An error analysis is also presented. Several examples are given to illustrate the convergence concept and error estimate.

1. Introduction

Many steady axisymmetrical diffusion problems [1–9] in science and engineering can be modeled by a class of two-point boundary value problems (BVPs)

$$y'' + \frac{m}{x}y' + f(x, y) = 0, \quad 0 < x \le 1 \tag{1}$$

$$y(0) = g(y(0), y'(0)), \quad m = 0 \quad (\text{or} \quad y'(0) = 0, m > 0) \quad y(1) = h(y(1), y'(1)), \tag{2}$$

where $g(r, s)$ and $h(r, s)$ are continuous. For the case $h = \alpha$, α is a real constant, the existence and uniqueness of solutions to the problem (1)-(2) have been established by Russell and Shampine [10] for $m = 1$ and 2 as well as by Chawla and Shivkumar [11] for $m \ge 1$.

The variational iteration method (VIM) proposed by He [12–14] has been recognized to be a reliable and efficient algorithm for solving various delay differential equations (DDEs) [15,16], ordinary differential equations (ODEs) [17–21], partial differential equations (PDEs) [22–24], and nonlinear problems arising in engineering [25–28] without requiring linearization, discretization or perturbation. When applied to the fractional differential equations (FDEs), the classical VIM considers the terms of fractional derivatives as restricted variations in determining the general Lagrange multiplier [29–31]. Recently, some alternative approaches including fractional VIM [32,33], fractional complex transform [33,34] and Laplace transform [35–37] have been adopted to remove such restriction and identify the Lagrange multiplier in a more accurate or efficient way. Based on the local fractional calculus, Yang and his colleagues [38–40] presented a local fractional VIM to solve local FDEs. Other modified VIMs such as variational iteration-Padé method [22], variational iteration-Adomian method [17,41],

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VIM with an auxiliary parameter [42–44] and optimal VIM [45] were suggested to accelerate the convergent rate, improve the accuracy or lengthen the interval of convergence for the obtained iterative solution. For more details about the method and its applications, the reader is referred to the review articles [46,47] and the references therein.

Significant research efforts have been made over the past decade to study the convergence of VIM for differential equations. Tatari and Dehghan [48] discussed the convergence concept of VIM by using Banach's fixed point theorem. Based on the same idea, the convergence analysis of VIM has been established for differential-algebraic equations [49], integral equations [50], PDEs [23,24], FDEs [51,52] and nonlinear differential equations [53]. Practically, it is difficult to verify that the operator used in the fixed point theorem is a contractive mapping in advance. Another approach to prove the convergence of VIM is to show that the sequences of error estimate obtained for DDEs [15,16], ODEs [18,19] and multi-order FDEs [37,54] are majorized by the general term of a convergent series. For nonlinear equations, the proofs were incomplete since the sufficient condition ensuring the nonlinear terms were Lipschitz within its domain of definition was not proved. Recently, this method has been successfully applied to the diffusion problem (1)–(2) with $h = \alpha - \beta y'(1)$, $\beta \geq 0$ [55–57]. However, no convergence proof is provided in [56,57] or the proof given in [55] is not very convincing and maybe wrong.

All the above mentioned studies focused on the old version of the method, which is called the variational iteration algorithm-I by He and his colleagues [47,58]. Based on the results obtained from some frequently used ODEs, they also constructed the variational iteration algorithm-II. Recently, such algorithm has been used to solve boundary value problems [59] and differential-difference equations [60]. In this paper, a new approach is introduced to construct the variational iteration algorithm-II for diffusion problems. Based on this algorithm, we prove that the method converges when applied to two-point BVPs (1)–(2) if $f(x, y)$ is Lipschitz in y and continuous in a region $R = \{(x, y) : 0 \leq x \leq 1, a \leq y \leq b\}$, where a and b are real constants. A formula for estimating the maximum error is also presented. Several examples are given to elucidate the convergence concept and error estimate.

2. Variational iteration algorithm

According to the variational iteration method [47,58], the correction functional (variational iteration algorithm-I) for Eq. (1) can be constructed as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s, x)\left[y_n''(s) + \frac{m}{s}y_n'(s) + f(s, y_n(s))\right]\mathrm{d}s \tag{3}$$

where $\lambda(s; x)$ is a general Lagrange multiplier derived as [55,57]

$$\lambda(s; x) = \begin{cases} s\ln\left(\dfrac{s}{x}\right), & m = 1 \\ \dfrac{s(s^{m-1} - x^{m-1})}{(m-1)x^m - 1}, & 0 \leq m \neq 1 \end{cases} \tag{4}$$

which is the optimal multiplier since only the term $f(x, y)$ in Eq. (1) is considered as a restricted variation. The derivatives of this Lagrange multiplier also satisfy

$$\begin{aligned} &\lambda(s = x, x) = 0 \\ &\left.\frac{\partial \lambda(s, x)}{\partial x}\right|_{s=x} = -1 \\ &\frac{\partial^2 \lambda(s, x)}{\partial x^2} + \frac{m}{x}\frac{\partial \lambda(s, x)}{\partial x} = 0 \end{aligned} \tag{5}$$

By differentiating Eq. (3) with respect to x and using the properties in Eq. (5), it is easy to prove $y_{n+1}(0) = y_0(0), y_{n+1}'(0) = y_0'(0)$ and

$$y_{n+1}''(x) + \frac{m}{x}y_{n+1}'(x) + f(x, y_n(x)) = 0, \quad n \geq 0 \tag{6}$$

Let the initial approximation be $y_0(x) = y(0) + y'(0)x$, then integrate the above equation twice to obtain the variational iteration algorithm-II as

$$\begin{aligned} y_{n+1}(x) &= y_0(x) - \int_0^x t^{-m}\int_0^t s^m f(s, y_n(s))\mathrm{d}s\mathrm{d}t \\ &= y_0(x) - \int_0^x \left(\int_s^x \frac{s^m}{t^m}\mathrm{d}t\right) f(s, y_n(s))\mathrm{d}s \\ &= y_0(x) + \int_0^x \lambda(s; x) f(s, y_n(s))\mathrm{d}s, \quad n \geq 0 \end{aligned} \tag{7}$$