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A biharmonic approach for the global stability analysis of 2D incompressible viscous flows

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ABSTRACT

This manuscript embodies an investigation of stabilities in flows governed by the incompressible Navier–Stokes equations with the recently developed compact scheme by Kalita and Gupta (2010) which was derived by using the biharmonic formulation of the 2-D incompressible Navier–Stokes equations. In the current work, we globally analyze the flow stability utilizing this transient ψ–v approach. Critical parameters are found by constructing a generalized eigenvalue problem resulting from the discretization of the stability equations through the above mentioned scheme. This approach is seen to drastically reduce the CPU time for finding the critical eigenvalues. Three fluid flow problems, namely, the square lid-driven cavity, the two sided cross lid-driven cavity and the flow past an inclined square cylinder have been chosen as test cases. For the square cavity and the flow past an inclined square cylinder, critical parameters found by us are in excellent agreement with the established results while for the two-sided cross cavity, its global stability analysis has been carried out for the first time and new results are obtained. The critical parameters are also reconfirmed by phase plane and spectral density analysis.

1. Introduction

Transition in fluid flows are of great practical importance and as such has generated tremendous interest amongst theoretical, experimental and computational fluid dynamists. One may cite the classical example of laminar to turbulent flow in a circular pipe, flow between concentric rotating cylinders, mean flows in turbulent buoyancy driven convection, sudden oscillatory behavior of plasma flows in Tokamak, transition in ocean circulation in Atlantic etc. The transition or the loss of stability is due to certain changes in the parameters in the system and several approaches exist for finding these critical parameters that determines the threshold regime of transition.

The Navier–Stokes (N–S) equations, which are the governing equations for incompressible viscous flows are the mainstay of capturing the transition in the flow. For majority of the fluid flow situations, the N–S equations do not possess analytical solution and numerical computation is the only way of solving them. In all the frameworks for solving the N–S equations, namely, the finite volume, finite element, finite difference or Lattice Boltzman, one starts with an initial condition. In the next step, the governing equations are integrated and the long time behavior of the quantities of interest are studied. In order to determine the transitional behavior and critical conditions, flow parameters are gradually changed, and the transient and asymptotic behavior are carefully examined from the computed solution. This paves the way for finding the transition

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between equilibrium behavior, either steady or time-dependent. The term Hopf bifurcation refers to the local birth or death of a periodic solution from an equilibrium point as a parameter crosses a critical value. In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearized flow at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher. When the real parts of the eigenvalues are negative the fixed point is a stable focus. When they cross zero and become positive the fixed point becomes an unstable focus, with orbits spiraling out. But this change of stability is a local change and the phase portrait sufficiently far from the fixed point will be qualitatively unaffected. Due to Hopf Bifurcation, a Spiral Point or focus switches from stable to unstable (or vice versa) and a periodic solution appears.

There exists several methods for determining the bifurcation point in transitional flow. The most popular amongst them are the Lyapunov Stability Theorems [1], using Stuart Landau equations [2,3] and by solving the generalized two-dimensional eigenvalue problem obtained from the linearized unsteady equations [4,5]; in the current study, we have opted for the last method. An excellent overview of the methods and approaches for studying global stability can be found in the recent work of Dijkstra et al. [6]. Note that all such studies for the 2D incompressible viscous flows have utilized either the streamfunction–vorticity (ψ–ω) or the primitive variable form of the N–S equations. The ψ–ω approach was seen to yield very accurate results for the type of flow considered. However, the presence of two variables (namely ψ and ω) invariably resulted in a much costlier matrix vector multiplication in the generalized eigenvalue problem, apart from the usual computational time in obtaining the steady state solutions for the flow variables considered. The primitive variable formulation is even more time consuming where one has to deal with three variables now, namely, the horizontal and the vertical velocities and the pressure.

Sanchez and his co-workers [7,8] were probably the first ones to utilize the pure-streamfunction (biharmonic [9]) formulation of the N–S equation to carry out global stability analysis of 2D incompressible fluid flow. They performed a linear stability analysis of the flow of an incompressible fluid in a completely filled cylinder with bottom and side walls rotating at constant angular speed and driven by the differential rotation of its top lid with an angular speed. They utilized a continuation method [10] with a predictor–corrector approach to discretize the steady-state equation; while for the unsteady part, they used a second order backward time differencing and second order space differencing with implicit treatment for the nonlinear terms. In the current work, we use a recently developed compact transient ψ–v finite difference scheme [11] to numerically solve this pure streamfunction formulation where the steady part of the stability equation is handled by the steady-state form of the same scheme. As would be seen later, this approach drastically reduces the CPU time for computing the eigenvalue and the corresponding eigenvectors for the generalized eigenvalue problem. In the following, we provide a brief description of the biharmonic formulation of the N–S equations and the finite different scheme employed by us to discretize them.

Recall the two-dimensional incompressible Navier–Stokes equations in the traditional primitive variable (velocity–pressure) formulation:

$$\rho(u_t + uu_x + vu_y) = -p_x + \mu(u_{xx} + u_{yy}), \tag{1}$$

$$\rho(v_t + uv_x + vv_y) = -p_y + \mu(v_{xx} + v_{yy}), \tag{2}$$

$$u_x + v_y = 0. \tag{3}$$

where u and v are the velocities along x- and y-directions respectively, t is time, p is the pressure, ρ is the density and μ is the viscosity of the fluid.

Despite accurately representing the fluid flow phenomena, the direct solution of this formulation traditionally has been difficult to obtain due to the pressure term in Eqs. (1) and (2). Therefore in order to avoid handling the pressure variable, an alternative formulation using streamfunction and vorticity has been used for several decades. This alternative formulation introduces the streamfunction ψ and vorticity ω, and subsequently Eqs. (1)–(3) in non-dimensional form reduce to:

$$\psi_{xx} + \psi_{yy} = -\omega(x, y). \tag{4}$$

$$\omega_t + u\omega_x + v\omega_y = \frac{1}{Re}(\omega_{xx} + \omega_{yy}), \tag{5}$$

where $(x, y) \in \Omega(\subset \mathbb{R}^2)$, Re is the non-dimensional Reynolds number defined by $Re = \frac{UL}{\nu}$ with U and L being some characteristic velocity and characteristic length, respectively. The non-dimensional velocity components are defined as:

$$u(x, y) = \psi_y \text{ and } v(x, y) = -\psi_x. \tag{6}$$

This ψ–ω formulation has been very successful and it has been widely used by researchers over the last several decades to validate newly established numerical methods for the numerical simulations of a variety of fluid flow problems. The major difficulty with this formulation lies in the specification of vorticity values at the no-slip boundaries. The vorticity ω, which can be defined through the Poisson equation $-\omega = \psi_{xx} + \psi_{yy}$, needs to be solved discretely on the boundaries so that the boundary values of the vorticity can be specified for the vorticity transport equation when this formulation is employed.