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A central scheme for advecting scalars by velocity fields obtained from Finite Volume multiphase incompressible solvers

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ABSTRACT

This paper presents an extension of the central scheme of Kurganov and Tadmor (2000) to work with solvers based on conservative face fluxes, which are usual in the solution of incompressible flows by the Finite Volume Method. The proposed scheme retains the desirable properties of simplicity, low numerical viscosity and multidimensionality, and it works on general non-staggered polyhedral meshes. It is applied within a mixture multiphase solver to discretize the mass conservation equation of one of the phases. A series of cases are solved which show that the proposed extension is significantly more robust and monotonicity-preserving than the straightforward application of the Kurganov–Tadmor scheme.

1. Introduction

Central schemes have attracted much attention due to their simplicity and effectiveness to solve hyperbolic partial differential equations. The principal advantage is that no Riemann problems have to be solved either exactly or using Godunov methods which require the analysis of upwind directions on the Riemann fan.

Starting from the seminal work of Lax [1] with the celebrated Lax–Friedrichs (LxF) scheme, central schemes have been improved in accuracy in space and time discretization. The first extension was due to Nessyahu and Tadmor (NT) [2] who obtained a second order generalization of the LxF scheme, decreasing the numerical viscosity from $\mathcal{O}(\Delta x^2/\Delta t)$ to $\mathcal{O}(\Delta x^4/\Delta t)$. The key concept was to replace the piecewise constant reconstruction of the solution by a piecewise linear representation with limited slopes [3]. This scheme has two main drawbacks: (1) it requires the use of staggered meshes and (2) the numerical viscosity increases for small time steps. Kurganov and Tadmor (KT) [4] removed these requirements and presented a second order central scheme with numerical viscosity of $\mathcal{O}(\Delta x^4)$ which accepts a semi-discrete formulation in a non-staggered grid.

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A remarkable characteristic of the KT is the direct extension to multidimensions, this topic was addressed by Kurganov and Tadmor in the presentation of the scheme. Thus, it is extended from the one dimensional case to more dimensions using the so-called *dimension-by-dimension* formulation. It implies the splitting of the original hyperbolic equation defining fluxes by the Cartesian directions. Truly multidimensional extensions were presented later, for the case of structured and triangular two dimensional meshes, Kurganov and Petrova [5,6] achieved such extensions using polynomial interpolants. Even though more improvements were done following the same line [7,8] none of them is capable to manage truly polyhedral collocated meshes. In this context, Greenshields et al. [9] made a valuable contribution treating arbitrary polyhedral meshes using a *face-by-face* reconstruction, recalling the original concepts of one dimensional central schemes of Kurganov and Tadmor but applied in the face normal direction.

Application of central schemes to single phase flows started considering the compressible case, in which the whole equation system is hyperbolic. Extension to multiphase flows also considered hyperbolic problems, such as those obtained by linking the density of each phase to the local pressure [10–13].

Central schemes were adapted for single phase incompressible flows by Kupferman [14], and have also been applied to viscoelastic liquids [15–18]. These contributions addressed particularly the solution of the momentum equation, the incidence of wave directions and capturing was not discussed. The schemes used were derived from the NT method in staggered structured meshes and were implemented by the Finite Difference Method. The application of KT schemes was later extended to multiphase porous media flow by Abreu et al. [19], Furtado et al. [3], and Pereira and Rahunanthan [20] with a special treatment of the coupling between the incompressibility restriction and the central advection scheme used to solve the Buckley–Leverett equation for the concentration. Their method was introduced considering Cartesian rectangular meshes with Raviart–Thomas finite elements for the computation of pressure and velocity. A key concept in the success of the method of Abreu et al. is the use of divergence-free fluxes in the advection of the concentration variables.

In this article an unstructured version of the method of Abreu et al. is introduced, which is capable to solve hyperbolic equations with a conservative flux given at faces and obtained in a pressure–velocity coupling loop (PISO) [21,22] and a nonlinear flux calculated by an explicit flux function (KTcFlux, Kurganov and Tadmor with conservative flux). The computation of the concentration fluxes uses the divergence-free velocities at faces obtained from the PISO algorithm together with the face-by-face reconstruction of concentration values proposed by Greenshields et al. The resulting scheme is an efficient cell-centered finite volume method for arbitrary meshes, that is locally and globally conservative and allows for capturing the shocks, rarefactions and compound waves which are usual in the solution of the multiphase systems [23].

The manuscript is organized as follows: the Section 1 represents the introduction. Next, in Section 2, some basic concepts of KT schemes are revisited and the KTcFlux is derived in semi-discrete form and extended to multidimensions. Section 3 presents an application of the present scheme to a multiphase mixture model. The obtained scheme is then applied to three examples which are presented in Section 4 and require the correct treatment of different kind of traveling waves. Finally, Section 5 presents the conclusions.

2. A Riemann-free solver with centered flux

2.1. The model hyperbolic equation

The application of central schemes is related to the non-linear hyperbolic model equation:

$$\frac{\partial}{\partial t}\vec{u}(\vec{x},t) + \vec{\nabla}\cdot\overline{\overline{\mathcal{F}}}(\vec{u}(\vec{x},t)) = 0, \tag{1}$$

where $\vec{u}$ is a vector of conserved quantities in a multidimensional space and $\overline{\overline{\mathcal{F}}}(\vec{u}(\vec{x},t))$ is a non-linear tensorial convection flux of the conserved quantities. These equations are commonly present in the description of problems of transport phenomena [24]. The discretization of Eq. (1) by the cell centered Finite Volume Method [22,25] using volume integrals and the Gauss theorem for the flux term leads to:

$$\int_{\Omega_j}\frac{\partial}{\partial t}\vec{u}(\vec{x},t)\,d\Omega + \int_{\Gamma_j}\overline{\overline{\mathcal{F}}}(\vec{u}(\vec{x},t))\cdot d\vec{\Gamma} \cong \frac{\partial}{\partial t}\bar{\vec{u}}_j V_\Omega + \sum_f \overline{\overline{\mathcal{F}}}(\bar{\vec{u}})_f\cdot\vec{S}_f = \frac{\partial}{\partial t}\bar{\vec{u}}_j V_{\Omega_j} + \sum_f \vec{\mathrm{F}}_f = 0, \tag{2}$$

where Ω_j is the volumetric domain of a given faceted finite volume j of the discretization, Γ_j is its boundary, V_{Ω_j} is the volume and $\bar{\vec{u}}_j$ and $(\bar{\vec{u}})_f$ the average values of $\vec{u}$ at cell centers and faces. In addition, f represents the index for each face of Ω_j, $\vec{S}_f$ the corresponding face area vector and $\vec{\mathrm{F}}_f = \overline{\overline{\mathcal{F}}}\cdot\vec{S}_f$ the total face flux.

2.2. One dimensional semi-discrete form

The derivation of the one dimensional semi-discrete form of the present scheme starts from the original Kurganov and Tadmor scheme, then it is worthy to recall some definitions (see Fig. 1):

- u: is the continuous function being integrated in time and space (scalar case of $\vec{u}$);
- $\bar{u}$: is a sliding average of u obtained by Eq. (3)
- $\tilde{u}$: is a piecewise linear approximation of u defined in Eq. (4).