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Modified homotopy perturbation method for optimal control problems using the Padé approximant

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ABSTRACT

In this study, we propose a hybrid method that combines the homotopy perturbation method (HPM) and Padé technique to obtain the approximate analytic solution of the Hamilton–Jacobi–Bellman equation. The truncated series solution for the HPM is suitable but only in a small domain when the exact solution is not obtained. To improve the accuracy and enlarge the convergence domain, the Padé technique is applied to the series solution for the HPM. Three examples are given to illustrate the applicability, simplicity, and efficiency of the proposed method. The results obtained are then compared with the exact solution and basic HPM. We demonstrate that this hybrid method provides an approximate analytic solution with higher accuracy than the classic HPM.

1. Introduction

The theory and application of optimal control are relevant to fields such as biomedicine, physics, economy, aerospace, chemical engineering, and robotics [1–5]. However, the optimal control of nonlinear systems is a difficult task, which has been studied for decades. Two main approaches are used to solve nonlinear optimal control problems. One approach is to address the problem directly and attempt to find the minimum of the objective functional. Indirect approaches are based on Pontryagin's maximum principle and they result in a two-point boundary value problem or Bellman's dynamic programming which leads to the Hamilton–Jacobi–Bellman (HJB) equation. The solution of the HJB equation is challenging due to its inherently nonlinear nature.

HJB equations have been solved using different techniques. An excellent review of developments in the solution of the HJB was provided by Beard et al. [6], who also considered the standard Galerkin approach (SGA). In the SGA, a sequence of generalized HJB equations are solved iteratively to obtain a sequence of approximations that approach the solution of the HJB equation; however, the proposed method may converge very slowly or even diverge. In [7], a polynomial approximation was proposed for solving the HJB equation. Another option is to use measure theory [8], which changes a nonlinear optimal control problem into a linear programming problem and this yields a piecewise constant control law. Fakharian and Beheshtia [9] used the Adomian decomposition method (ADM) to solve the HJB equation. The variational iteration method was applied to nonlinear quadratic optimal control problems in [10,11]. Saberi and Effati [12] proposed a novel computational approach for generating suboptimal solutions for a class of nonlinear optimal control problems based on the one-dimensional differential transform method and new polynomials called DT polynomials. In [13], the homotopy analysis method was employed to solve linear optimal control problems with a quadratic performance index. Several studies have

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proposed other techniques for solving linear and nonlinear optimal control problems [14–18], but it is usually difficult to obtain exact or analytical solutions for the HJB equation, and thus an approximate-analytic method is used to solve it.

In recent years, the homotopy perturbation method (HPM) was proposed by He [19,20] as an efficient and accurate approach for solving various linear and nonlinear equations. This method combines the traditional perturbation method and homotopy in topology, thereby transforming a difficult problem into a simple problem that is easily solved. In contrast to the traditional perturbation methods, the HPM does not require discretization or linearization and it does not depend on a small parameter that is difficult to find. This method successfully provides an analytical approximate solution to a wide range of linear and nonlinear problems in different fields.

He used HPM to solve the Duffing equation [20] and Blasius equation [21], and the concept was applied subsequently to solve nonlinear wave equations [22], boundary value problems [23,24], quadratic Riccati differential equations [25,26], integral equations [27,28], heat transfer analysis [29], two dimensional time-fractional wave equation [30], Boussinesq-like equations [31], two-dimensional Volterra–Fredholm integral equations [32], dynamic responses of fractionally damped mechanical system [33], Volterra–Fredholm integral equation [34], and many other problems [35–42]. In [43], a series solution was obtained using the HPM for the long porous slider problem where fluid is injected through the porous bottom. Recently, He [44] proposed an alternative approach to constructing the homotopy equation using an auxiliary term and a HPM with two expanding parameters, which are especially effective for a nonlinear equation with two nonlinear terms [45]. The effects of variable viscosity and thermal conductivity on the flow and heat transfer in a laminar liquid film were analyzed by Khan et al. using HPM [46]. In [47], a new modified version of the HPM and the ADM was proposed to solve a nonlinear ordinary differential equation arising in an MHD non-Newtonian fluid flow over a linear stretching sheet. In [48], a new modification of the HPM was applied to obtain an analytic approximation of stiff systems of ordinary differential equations.

A standard HPM was used to solve the HJB equation in previous studies [49,50] and Atangana et al. [51] proposed the homotopy decomposition method to solve the HJB equation, but these solutions are only suitable in a small region. Thus, it is necessary to increase the region of convergence for the approximate solution. Effati et al. [52] used a piecewise HPM (PHPM) to solve the HJB equation and obtain accurate approximate solutions for a large region, but this method is very sensitive to changes in the sample time (Δt) and it may be unstable for some Δt. Moreover, the solution can be complicated for larger intervals. In order to obtain greater accuracy and rapid convergence in the HPM solution, previous studies [53,54] proposed the HPM coupled with the Laplace transformation to solve nonlinear equations and a time-dependent drainage model, respectively. In [55], He's polynomials were coupled with the Laplace decomposition method to solve nonlinear oscillator differential equations. In [56], an auxiliary parameter method using Adomian polynomials and Laplace transformation was introduced to obtain the solutions of differential-difference equations in a rapid convergent series in closed form.

In the present study, we combine the HPM and Padé techniques to solve the HJB equation, and we then demonstrate that the application of Padé approximants to the HPM truncated series solution improves the rate of convergence, increases the accuracy, and enlarges the region of convergence. Several examples are given to illustrate the superior accuracy and convergence of the proposed method compared with the standard HPM.

The remainder of this paper is organized as follows. In Section 2, we present the nonlinear optimal control problem. In Section 3, we explain the HPM. In Section 4, we describe the Padé approximation. In Section 5, we apply the proposed method to a variety of examples to demonstrate the efficiency of the proposed method. In Section 6, we give our conclusions.

2. Nonlinear optimal control

In this section, we provide a brief description of nonlinear optimal control. First, consider the following nonlinear control system:

$$\dot{x}(t) = a(x(t), u(t), t), \tag{1}$$

where $x(t)$ is a state vector and $u(t)$ is a control signal. The aim is control of the system and determining the optimal control law that minimizes the following cost function

$$J = h(x(t_f, t_f) + \int_{t_0}^{t_f} g(x(\tau), u(\tau), \tau)d\tau \tag{2}$$

where h and g are arbitrary convex functions, and t_f is the final time of system operation. A new variable is introduced using the dynamic programming approach as follows.

$$J(x(t), t, u(t)) = h(x(t_f, t_f) + \int_{t_0}^{t_f} g(x(\tau), u(\tau), \tau)d\tau, \qquad t_0 < t < t_f, t < \tau < t_f \tag{3}$$

Suppose that

$$V(x(t), t) = J^*(x(t), t) = \min_{u(\tau), t < \tau < t_f} \left\{ h(x(t_f, t_f) + \int_{t_0}^{t_f} g(x(\tau), u(\tau), \tau)d\tau \right\}. \tag{4}$$