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Beta Sarhan–Zaindin modified Weibull distribution

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ABSTRACT

We introduce a new distribution, so-called beta Sarhan–Zaindin modified Weibull (BSZMW) distribution, which extends a number of recent distributions, among which the modified-Weibull, the exponentiated modified-Weibull, beta Weibull and beta linear failure rate distributions. Various structural properties of the distribution are obtained (sometimes in terms of Meijer's *G*-function), such as the moments, moment generating function, conditional moments, mean deviations, entropy, order statistics, mean and variance of the (reversed) residual life and maximum likelihood estimators as well as the observed information matrix. The distribution exhibits a wide range of shapes with varying skewness and assumes all possible forms of hazard rate function. The BSZMW distribution along with other distributions are fitted to two sets of data, arising in hydrology and in meteorology. It is shown that, the distribution has a superior performance among the compared distributions as evidenced by some goodness-of-fit tests. As well, some statistical functions associated with these data such as the return level and mean deviation about the return level are obtained.

1. Introduction

The Weibull distribution is a very popular distribution for modeling and analyzing many hydrological and meteorological data sets as well as lifetime data with monotonic failure rates. The Weibull distribution cannot be utilized to model some types of such data because it cannot accommodate to marked skewness or bathtub and upside-down bathtub hazard rate shapes. To overcome such shortcomings, various generalizations of the classical Weibull distribution have been investigated by several authors in the recent years; among them, the extended flexible Weibull distribution [1], the generalized modified Weibull distribution [2], the exponentiated Weibull distribution [3], the additive Weibull distribution [4] and the extended Weibull distribution [5]. Also, Sarhan and Zaindin [6] introduced the modified-Weibull (MW) distribution having three parameters $\lambda > 0, \beta > 0$ and $k > 0$, with the cumulative distribution function (cdf) and probability density function (pdf)

$$G_{\lambda,\beta,k}(x) = 1 - e^{-\lambda x - \beta x^k}, \tag{1}$$

and

$$g_{\lambda,\beta,k}(x) = (\lambda + \beta k x^{k-1})\, e^{-\lambda x - \beta x^k}, \quad x > 0, \tag{2}$$

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Table 1
Some special cases of the BSZMW distribution

Parameteric values in (6)	Sub-model
$\rho = 1$	Exponentiated modified-Weibull (λ, β, k, α) [14]
$\alpha = \rho = 1$	Modified-Weibull (λ, β, k) [6]
$\lambda = 0$ and $\beta = \beta^k$	Beta-Weibull distribution (β, k, α, ρ) [15]
$k = 1$ and $\beta = 0$	Beta-exponential distribution (λ, α, ρ) [9]
$\lambda = 0, \beta = \beta/2$ and $k = 2$	Beta-Rayleigh distribution (β, α, ρ) [16]
$\beta = \beta/2$ and $k = 2$	Beta linear failure rate (λ, β, α, ρ) [17]

respectively. It is worth noting that the MW distribution contains both the exponential ($\beta = 0$) and Weibull ($\lambda = 0$) distributions.

Other generalizations of the Weibull distribution can be induced from the generalized beta-family of distributions generated by beta random variables (Jones [7]) as

$$F(x) = \frac{B(G(x); \alpha, \rho)}{B(\alpha,\ \rho)}, \tag{3}$$

where the beta function $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $\Re(a) > 0$, $\Re(b) > 0$, the incomplete beta function $B(z; a, b) = \int_0^z t^{a-1}(1-t)^{b-1}dt$ and $G(x)$ denotes the cdf of a baseline distribution and $\alpha > 0$, $\rho > 0$. The pdf corresponding to (3) can be written in the form

$$f(x) = \frac{1}{B(\alpha,\ \rho)} G(x)^{\alpha-1}\{1 - G(x)\}^{\rho-1} g(x), \tag{4}$$

where $g(x) = \mathrm{d}G(x)/\mathrm{d}x$ is the density of the baseline distribution. Jones [7] demonstrated that the pdf in (4) is a generalization of the pdf of the order statistic of a random sample taken from the cdf $G(x)$ and investigated some properties of such a family. This family of generalized distributions has received much attention over the last few years during which several forms of beta-compounded distributions have been investigated. For instance, Eugene et al. [8] introduced the beta normal distribution by taking $G(x)$ in (3) to be the cdf of the normal distribution and obtained some of its properties; Nadarajah and Kotz [9,10] proposed both the beta-exponential and the beta-Gumbel distributions and studied some of their properties; Silva et al. [11] worked on the beta modified Weibull distribution, based on the modified Weibull distribution proposed by Lai et al. [12], and discussed many of its properties along with estimation issues; and Cordeiro et al. [13] discussed the beta-exponentiated Weibull distribution and some of its properties.

In this paper, we introduce a new distribution having five parameters which is based on the MW distribution and (3), so-called Beta Sarhan-Zaindin modified Weibull (BSZMW) distribution, which is a generalization for the MW distribution. Our motivation for introducing the BSZMW distribution is due to the simple analytic expressions of $G_{\lambda,\ \beta,\ k}(x)$ and $g_{\lambda,\ \beta,\ k}(x)$ for the MW distribution. Accordingly, we obtain a distribution having a tractable and flexible density, which is not the case for the beta-exponential and the beta-exponentiated Weibull distribution among others. Moreover, it can be represented as a weighted infinite linear combination of MW distributions, and thus many properties of the BSZMW distribution result from those associated with the corresponding ones of the MW distribution. Moreover, the distribution contains several distributions as special cases, several of them being listed in Table 1. Additionally, the distribution provides a wide range of shapes with varying skewness, varied tail weights and shifting modes based on its additional parameters. It also accommodates most forms of hazard rates that are encountered in a variety of real-life problems, and hence it affords flexibility in fitting data arising in a wide variety of natural applications. In light of all these considerations, it is hoped that this distribution will accommodate a wider range of applications in hydrology and metrology as well as in other areas of research.

We now introduce the BSZMW distribution by taking $G(x)$ in (3) to be the cdf (1) of the MW distribution. Accordingly, the cdf of the BSZMW distribution is

$$F(x) = \frac{B(1 - \mathrm{e}^{-(\lambda x + \beta x^k)};\ \alpha,\ \rho)}{B(\alpha,\ \rho)}. \tag{5}$$

The BSZMW density function can be obtained from Eqs. (1), (2) and (4) as

$$f(x) = \frac{(\lambda + \beta k x^{k-1})\mathrm{e}^{-\rho\lambda x - \rho\beta x^k}(1 - \mathrm{e}^{-\lambda x - \beta x^k})^{\alpha-1}}{B(\alpha, \rho)}, \quad x > 0. \tag{6}$$

Beside interpreting (6) as a density generated by beta random variables, it can also be interpreted in another manner as a weighted density function with weight function $w(x) = \mathrm{e}^{-(\rho-1)\lambda x - (\rho-1)\beta x^k}(1 - \mathrm{e}^{-\lambda x - \beta x^k})^{\alpha-1}$ and normalizing constant $B(\alpha, \rho) = E[w(x)]$. An extra motivation for considering (6) is that it constitutes a generalization of the pdf of the order statistic of a random sample taken from cdf (1), and hence a reasonable interpretation of the parameters α and ρ may be that α acts as the order of order statistics with cdf (1), ρ being the sample size of those order statistics minus α.