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Numerical solution of time-fractional nonlinear PDEs with proportional delays by homotopy perturbation method

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ABSTRACT

In this paper, homotopy perturbation method (HPM) is applied to solve fractional partial differential equations (PDEs) with proportional delay in t and shrinking in x. The method do not require linearization or small perturbation. The fractional derivatives are taken in the Caputo sense. The present method performs extremely well in terms of efficiency and simplicity. Numerical results for different particular cases of α are presented graphically.

1. Introduction

In recent years, many significant phenomena in fluid mechanics, electrical networks, signal processing, diffusion, reaction processes and other fields of science and engineering can be modeled with the aid of fractional derivatives [1,2]. In general, it is hard to obtain the exact solution to large class of fractional differential equations, so approximate and numerical methods must be applied. A lot of vigorous methods have been constructed for solving fractional differential equations. These methods include the Laplace decomposition method [3], the reproducing kernel Hilbert space method [4], the generalized differential transform method [5], the homotopy analysis method [6], the variational iteration method [7], the local fractional variational iteration method [8], the fractional complex transform [9] and Adomian's decomposition method [10].

Partial functional-differential equations with proportional delays represent a particular class of delay partial differential equation. This type equations arise from various applications such as in biology, medicine, population ecology, control systems and climate models [11]. Their independent variables are time t and one or more dimensional variable x, which usually represents position in space or size of cells, maturation level and so on. The solutions may represent voltage, temperature, densities of different particles, such as chemicals, cells, etc. There is so little experience with numerical methods for solving delay partial differential equations. Zubik-Kowal [12] applied the Chebyshev pseudospectral method for linear differential and differential-functional parabolic equations. Zubik-Kowal and Jackiewicz [13] used the spectral collocation and the waveform relaxation methods for solving nonlinear delay partial differential equations. Abazari and Ganji [14] proposed two dimensional differential transform method and its reduced form to obtain the solution of partial differential equations with proportional delay. Tanthanuch [15] utilized the group analysis method for nonhomogeneous inviscid Burgers equation with

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delay. Polyanin and Zhurov [16] suggested functional constraints method for constructing exact solutions to nonlinear delay reaction–diffusion equations.

In this paper, we consider the following time-fractional partial differential equations with proportional delays [14]:

$$ {}_a^C D_t^{\alpha} u(x,t) = f\left(x, t, u(p_0 x, q_0 t), \frac{\partial}{\partial x} u(p_1 x, q_1 t), \ \ldots, \frac{\partial^n}{\partial x^n} u(p_n x, q_n t)\right), \tag{1} $$

with the initial condition,

$$ u^{(k)}(x,0) = g_k(x), \tag{2} $$

where p_i, $q_j \in (0, 1)$ for $i, j \in N$, $g_k(x)$ is a given initial value, and f is the partial differential operator. A large variety of delay PDE models can be found in [11].

One important example of our model is Korteweg–de Vries (KdV) equation which arise in the research of shallow water waves as follow:

$$ {}_a^C D_t^{\alpha} u(x,t) + buu_x(p_0 x, q_0 t) + u_{xxx}(p_1 x, q_1 t) = 0, \quad 0 < \alpha \le 1, $$

where b is a constant.

Another well-known model is the time-fractional nonlinear Klein–Gordon equation with proportional delay which comes from quantum field theory and describes nonlinear wave interaction:

$$ {}_a^C D_t^{\alpha} u(x,t) = u_{xx}(p_0 x, q_0 t) - bu(p_1 x, q_1 t) - F(u(p_2 x, q_2 t)) + h(x,t), \quad 1 < \alpha \le 2, $$

where b is a constant, $h(x, t)$ is a known analytical function or source term and $F(u(p_2 x, q_2 t))$ is a nonlinear function of $u(x, t)$.

Homotopy perturbation method (HPM) provides an analytical approximate solution for linear and nonlinear differential equations. This method is a combination of the traditional perturbation method and homotopy concept in topology. HPM can give the approximate or exact solutions without discretization, linearization, transformation or small perturbation. The homotopy perturbation method firstly proposed by He [17,18]. This technique has been proved by many authors to be an efficient mathematical tool for solving various kinds of problems such as algebraic equations [19], nonlinear oscillator differential equations [20], delay differential equations [21], non-linear fredholm integral equation [22], wave and nonlinear diffusion equations [23], seventh-order generalized KdV equation [24], fractional predator–prey system [25], time-fractional Fornberg–Whitham equations [26], two dimensional time-fractional wave equation [27], space-time fractional solidification in a finite slab [28].

There are many different definition of fractional order derivatives. Caputo derivative is defined only for differentiable functions, while function can be a continuous but not necessarily differentiable. Riemann–Liouville (R–L) definition can be used any functions that are continuous but not differentiable anywhere, however, R–L derivative of constant is not equal to zero. To overcome the shortcomings, Jumarie [29] offered the modified R–L derivative where function is a continuous but not necessarily differentiable. It is important to say that R–L derivatives of non-integer orders cannot satisfy the Leibniz and chain rules in general [30,31]. The local fractional derivative is suggested by Yang and companions [32,33]. Recently, various type of He's fractional derivative are given in [34]. Fractional derivative is valid also for discontinuous problem. For detail please refer to [35].

This paper is organized as follows: some basic definitions of fractional calculus are introduced in Section 2. Application of homotopy perturbation method will be illustrated in Section 3. Section 4 is devoted to convergence analysis the homotopy perturbation method and the error estimates of this method. Applications are shown in Section 5, when the fractional partial differential equations with proportional delay are successfully solved by the presented algorithm. Section 6 is devoted to numerical results and discussion. Finally, the work will be concluded at Section 7.

2. Fractional calculus

Fractional calculus unifies and generalizes the notions of integer-order differentiation and n-fold integration [1,2]. We give some basic definitions and properties of fractional calculus theory which shall be used.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_m$, $m \in N$. The Riemann–Liouville fractional integral operator is defined as follows:

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \ge 0$, of a function $f \in C_\mu$, $\mu \ge -1$, is defined as:

$$ J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > 0, $$

$$ J^0 f(x) = f(x). $$

Properties of the operator J^α can be found in [1,2] and we mention only the following: For $f \in C_\mu$, $\mu \ge -1$, α, $\beta \ge 0$, and $\gamma \ge -1$: