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Dynamical behaviors, circuit realization, chaos control, and synchronization of a new fractional order hyperchaotic system

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ABSTRACT

In this study, we introduce a new fractional order hyperchaotic system. We prove the existence and uniqueness of the solution to the proposed system. We also study the stability of the system's equilibrium points and the continuous dependence of the system's solution on the initial conditions. The dynamical behavior of the system is explored and a circuit implementation of the fractional order system is proposed. We examine the effects of fractional order derivatives on chaos control for the proposed system. examined. Finally, synchronization is achieved between two fractional order hyperchaotic systems using a linear feedback controller and a time-delayed feedback controller. Numerical simulations are presented that verify the theoretical analysis.

1. Introduction

Fractional calculus is the field of mathematical analysis that deals with the investigation and application of integrals and derivatives of arbitrary order. Fractional order calculus has attracted much interest and importance in various fields of physics and engineering, where the number of applications of fractional calculus has increased rapidly in recent years (see [1,2], and the references therein). Employing the notion of fractional-order when modeling and simulating systems may be a more realistic approach because real phenomena are generally fractional [3]. Thus, fractional order calculus-based modeling tools may allow us to describe and model real processes more accurately than integer order methods.

Based on the definitions of fractional order derivatives and integrals, it can be observed that the use of fractional order differential equation-based modeling tools yields more accurate descriptions as well as providing deeper insights into the natural and man-made processes that underlie long-range memory behavior in a more accurate manner than integer order models.

For example, capacitors are crucial elements of integrated circuits and they are used extensively in many electronic systems. However, Jonscher [4] demonstrated that the ideal capacitor cannot exist in nature because an impedance of the form $1/(j\omega C)$ would violate causality. Thus, the need to find more realistic models of capacitors has led to the use of fractional calculus as a modeling tool.

The assumption of a fractional order derivative model for the current–voltage behavior of a capacitor based on the assumption that the capacitor "remembers" voltages to which it was subjected previously was proposed by Westerlund and Ekstam in [5], who solved this problem successfully using the Riemann–Liouville fractional order derivative.

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Another example from electronic circuits is the memristor, which is an element that relates the magnetic flux between its terminals via the electric charge that passes through it. Thus, it is considered to possess memory because the magnetic flux and the electric charge at any time moment t are integrals of the voltage across the memristor and the current that passed through it, respectively, from the initial value at time t_0 until time t. Therefore, a realistic model of a memristor has been postulated as a fractional order model to capture the dynamical behavior of the memristor.

Fractional calculus has been applied to various engineering problems related to electronic circuits, chaotic and hyperchaotic systems such as fractional order modeling of RC circuits, LC circuits, second order filters, sinusoidal oscillators, memristor elements, ADVP circuits, Liu chaotic and hyperchaotic systems, chaos control, and the synchronization of fractional order chaotic systems (see [6–15]).

In various area of engineering, physics, biology, chemistry, and economics, we may encounter systems that undergo spatial and temporal evolution [16] and [17]. The study of dynamical systems is a useful tool for modeling, understanding, and analyzing these phenomena. Chaos is an important and fascinating behavior that exists in some dynamical systems.

A chaotic dynamical system is characterized by its sensitive dependence on the initial conditions and the presence of positive Lyapunov exponents in its attractor. If the system's attractor has more than one positive Lyapunov exponent, it is called a hyperchaotic system. Hyperchaotic systems are more unpredictable and random than simple chaotic systems. Thus, hyperchaos is preferred in many applications including secure communications, chaos-based image encryption, and cryptography. The applications of dynamical systems and chaos involve mathematical biology, financial systems, chaos control and synchronization, electronic circuits, secure communications, image encryption, cryptography, and neuroscience research [18–29].

Based on the observations mentioned above, in this study, we modify the system in [18] by considering fractional order models of its components. We study the existence, uniqueness, and continuous dependence on the initial conditions for the solution of the system. We examine the stability of the equilibrium points of the proposed hyperchaotic system. We also explore the dynamics of the hyperchaotic system using bifurcation diagrams and phase portraits of the system's state variables.

We propose a circuit implementation to realize the fractional order hyperchaotic system. To the best of our knowledge, we implement a circuit for some values of fractional order α that have not been described previously (e.g., see [7], [9], [30], and references therein). We present the results of simulations of the proposed circuit, which we compare with the results of numerical simulations to clarify the performance of the proposed circuit. We use the well known "Predict, Evaluate, Correct, Evaluate" (PECE) scheme ([31,32]) as a numerical method to perform the numerical simulations because simulating a fractional order system using the time domain methods is more reliable than frequency-based approximation ([1,33]).

We consider the effect of the fractional order derivative on controlling hyperchaos in the proposed system and we show that the fractional order hyperchaotic system is controlled only in the fractional-order case for selected values of the controllers. Finally, chaos synchronization between the drive and response systems is achieved using two techniques, i.e., linear control and time-delayed feedback control, where we show that synchronization can be achieved using only one controller based on time-delayed feedback control.

The remainder of this paper is organized as follows. The proposed system is introduced in Section 2. The existence and uniqueness of the proposed system's solution are studied in Section 3. The stability of the equilibrium points of the system is studied in Section 4. The continuous dependence of the system's solution on the initial conditions is derived in Section 5. Numerical simulations are presented in Section 6. The circuit implementation of the model is described in Section 7. We study chaos control and chaos synchronization of the hyperchaotic system in Sections 8 and 9, respectively. In Section 10, we give our conclusions and a general discussion of this study.

2. The proposed hyperchaotic fractional order system

Different definitions of fractional order differentiation and integration have appeared throughout the development of fractional order theory. The well-established definitions include the Cauchy integral formula, the Grünwald–Letnikov definition, the Riemann–Liouville definition, and the Caputo definition ([1,2]).

Let $L^1 = L^1[a, b]$, $0 \le a < b < \infty$, be the class of Lebesgue integrable functions on $[a, b]$.

Definition 1. The Riemann–Liouville definition of a fractional integral of the order $\alpha > 0$ for the function $f(t) \in L^1$, $t > 0$, is given by [2]

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{1}$$

where $\Gamma(\cdot)$ is Euler's Gamma function.

Definition 2. The Caputo definition of fractional derivatives of order α, $0 < \alpha < 1$ for the absolutely continuous function $f(t)$ is given by [2]

$$D^\alpha f(t) = I^{1-\alpha} \frac{df(t)}{dt} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{2}$$

The advantage of using the Caputo definition of fractional order derivatives when modeling practical systems is due to the fact that the initial conditions for the fractional-order differential equations with the Caputo derivatives are of the same form as those for the integer-order differential equations, and there are clear interpretations of the initial conditions for integer orders