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Legendre spectral element method for solving time fractional modified anomalous sub-diffusion equation

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A B S T R A C T

In this paper, an efficient numerical method is proposed for the solution of time fractional modified anomalous sub-diffusion equation. The proposed method is based on a finite difference scheme in time variable and Legendre spectral element method for space component. The fractional derivative of equation is described in the Riemann–Liouville sense. Firstly, for obtaining a semi-discrete scheme, the time fractional derivative of the mentioned equation has been discretized by integrating both sides of it. Secondly, we use the Legendre spectral element method for full discretization in one- and two-dimensional cases. In this approach the time fractional derivative of mentioned equation is approximated by a scheme of order $\mathcal{O}(\tau^{1+\gamma})$ for $0 < \gamma < 1$. We prove the stability and convergence of time discrete scheme using energy method, and show the time discrete scheme is convergent. Also, we propose an error estimate for the full discretization scheme. Numerical examples confirm the high accuracy and efficiency of the proposed numerical scheme.

1. Introduction

In recent years there has been a growing interest in the field of fractional calculus [1–3]. Fractional differential equations have attracted increasing attention because they have applications in various fields of science and engineering [75]. Many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order. Some of the most applications are given in the book of Oldham and Spanier [4], the book of Podlubny [3] and the papers of Metzler and Klafter [5], Bagley and Trovik [6]. Many considerable works on the theoretical analysis [7,8] have been carried on, but analytic solutions of most fractional differential equations cannot be obtained explicitly, so proposing new method to finding the numerical solution of these equations is of practical importance. There are several definitions of a fractional derivative of order $\alpha > 0$ [2,4,73]. The two most commonly used are the Riemann–Liouville and Caputo. The difference between the two definitions is in the order of evaluation. We start with recalling the essentials of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. Also we refer the interested reader to [9] for the numerical solution of fractional differential equations. Now, we give some basic definitions and properties of the fractional calculus theory.

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Definition 1 [3,4]**.** For $\mu \in \mathbb{R}$ and $x > 0$, a real function $f(x)$, is said to be in the space C_μ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and for $m \in \mathbb{N}$ it is said to be in the space C_μ^m if $f^m \in C_\mu$.

Definition 2 [3,4]**.** The Riemann–Liouville fractional integral operator of order $\alpha > 0$ for a function $f(x) \in C_\mu$, $\mu \geq -1$, is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > 0, \quad J^0 f(x) = f(x).$$

Also we have the following properties:

$$\bullet\, J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \quad \bullet\, J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \quad \bullet\, J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 3 [3,4]**.** If m be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as:

$$ {}_0^C D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{\partial^m u(x,s)}{\partial s^m} (t-s)^{m-\alpha-1} ds, & m-1 < \alpha < m, \quad m \in \mathbb{N}, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & m = \alpha. \end{cases} \tag{1.1}$$

Recently, Sokolov and Klafter [10–12] proposed a model for describing processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative to the following form [13,14]:

$$\frac{\partial u(x,t)}{\partial t} = \left(\upsilon_1 D_t^{1-\alpha} + \upsilon_2 D_t^{1-\beta}\right)\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}\right] + f(x,t),$$

where $0 < \alpha, \beta < 1$ and υ_1, υ_2 are positive constants. A possible application of this equation is in econophysics where there is an increasing interest in modeling using continuous time random walk (CTRWs) [15–17]. Authors of [18] studied the high order finite difference method for solving the reaction and anomalous-diffusion equations of the following form:

$$\frac{\partial u}{\partial t} = {}_0 D_t^{1-\gamma}\left[K_\gamma \frac{\partial^2 u}{\partial x^2} + C_\gamma u(x,t)\right] + f(x,t),$$

where ${}_0 D_t^{1-\gamma}$ t is the Riemann–Liouville time fractional partial derivative of order $1-\gamma$ and also K_γ and C_γ are positive constants.

In this paper, we consider the time fractional modified anomalous sub-diffusion equation:

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = \left(\upsilon_1 D_t^{1-\alpha} + \upsilon_2 D_t^{1-\beta}\right)\Delta u(x,y,t) + f(x,y,t), \\ 0 < t \leq T, \quad x \in \Omega = [a,b] \times [c,d], \end{cases} \tag{1.2}$$

with the boundary conditions:

$$u(x,y,t) = h(x,y,t), \quad (x,y) \in \partial\Omega, \tag{1.3}$$

and the initial condition:

$$u(x,y,0) = g(x,y), \quad (x,y) \in \Omega, \tag{1.4}$$

where $0 < \alpha, \beta < 1$, υ_1 and υ_2 are positive constants, Δ is Laplace operator and ${}_0 D_t^{1-\alpha} u(x,t)$ and ${}_0 D_t^{1-\beta} u(x,t)$ are the Rieman–Liouville fractional derivatives of orders $1-\alpha$ and $1-\beta$ defined by:

$${}_0 D_t^{1-\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\eta)}{(t-\eta)^{1-\alpha}} d\eta, \quad {}_0 D_t^{1-\beta} u(x,t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\eta)}{(t-\eta)^{1-\beta}} d\eta,$$

respectively. Liu et al. [19] proposed a semi-discrete approximation and a full discrete finite element approximation for the modified anomalous subdiffusion Eqs. (1.2)–(1.4) in a finite domain. They proved the stability and convergence of the proposed methods. Authors in [14] proposed a conditionally stable finite difference scheme for the solution of (1.2)–(1.4). They showed that the convergence order of the method is $\mathcal{O}(\tau + h^2)$ with energy method. The aim of [20] is to study the high order difference scheme for the solution of modified anomalous fractional sub-diffusion equation in which the convergence order of the method is $\mathcal{O}(\tau + h^4)$. Authors in [21] have designed the finite difference/element methods for a two-dimensional modified fractional diffusion equation. They have shown that both the semi-discrete and full discrete schemes are unconditionally stable and convergent. Langlands in [13] has given the solution of the modified equation on an infinite domain. In contrast to the solution of the traditional fractional diffusion equation, the solution of the modified equation requires a summation of Fox functions [22] instead of a single Fox function.