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Stability and well-posedness of a rate-dependent damage model for brittle materials based on crack mechanics

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ABSTRACT

This paper presents an investigation of the stability and well-posedness of a rate-dependent damage model for brittle materials. The model is based on the response of an ensemble of distributed microcracks under a general, three-dimensional state of stress. The stability and well-posedness of the model are studied by examining the behavior of dynamic perturbations to the steady-state solution of uniaxial-stress loading. It is shown that as a result of incorporating the strain-rate effect in the model, perturbations of all wave lengths remain bounded for finite times, making the problem well-posed. It is also shown that the corresponding rate-independent model is ill-posed in that perturbations grow unbounded with the wave number, even for finite times.

1. Introduction

Brittle and quasi-brittle materials are encountered in a wide range of applications, including ceramics in modern personnel and vehicle armor, crystals in polymer-bonded explosives, and concrete (a quasi-brittle material) in construction. In this work we consider only brittle materials (ceramics in particular) whose mechanical behavior is controlled by the response of cracks in the materials (in quasi-brittle materials other mechanisms, such as plastic deformation, can also play an important role). Brittle materials tend to exhibit high compressive strength but low tensile strength. In addition, at low temperatures or high strain rates some ductile materials can exhibit behavior that can be described as brittle in nature. As a result, modeling the damage and failure of brittle materials has become increasingly important in order to appropriately design structures containing brittle materials and to avoid catastrophic failures. Over the last 30 years, multiple models [1–30] have been developed to study the behavior of brittle materials under dynamic loading. For example, Dubé et al. [1] developed a rate-dependent damage model for concrete under dynamic loading. Zhang et al. [2] developed an anisotropic model for dynamic damage and fragmentation of rock under explosive loading. These and many other models are described in a recent review paper by Zhang and Zhao [3].

The models that have been developed range from simple empirical models to micromechanics-based models that provide more accurate descriptions of material responses. A successful material model needs to be capable of predicting the observed phenomena of the materials. One phenomenon in brittle materials that has been of a considerable challenge to predict is strain-softening, which occurs due to accumulation of damage, such as microcracks and voids. This softening response can cause the Initial-Boundary-Value Problem (IBVP) associated with the model to be mathematically ill-posed if the model does not take strain-rate or spatially nonlocal effects into account [4]. In this context ill-posedness of an IBVP means that some small differences

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in initial-boundary conditions can lead to grossly different solutions, even for a finite time. Practically speaking, this often leads to lack of convergence upon mesh refinement in numerical solutions of the Initial-Boundary-Value Problem. Even a well-posed model for brittle materials tends to exhibit instability. That is, solutions to problems with slightly different initial conditions diverge over time, but the differences are bounded for finite times. If the model accurately captures the material behavior, then this can be due to the unstable nature of the brittle material itself. In the current work we will examine the well-posedness and stability of a particular model for damage in brittle materials, the Dominant Crack Algorithm (DCA) model.

It is the goal of this work to show that the Dominant Crack Algorithm (DCA) model for damage in brittle materials is well-posed. Furthermore, it captures the instability of the underlying material behavior, resulting in the mathematical instability of the steady-state solution.

The DCA model, developed by Zuo et al. [5], is a micromechanical model that accounts for strain-rate dependence through dynamic crack growth. Its theoretical formulation is closely related to that of the Statistical Crack Mechanics model [6] and of the Isotropic Statistical Crack Mechanics model [7]. The DCA model incorporates anisotropy of damage through the orientation of the dominant crack. Recently in [8] the model has been expanded to include plasticity for quasi-brittle materials, though that version of the model is not examined here. The DCA model is relatively simple to implement from a computational perspective, yet sufficiently robust to capture softening behavior of brittle materials due to crack growth. The model has been implemented into structural analysis codes and used in engineering applications, hence it seems to be worthwhile to determine whether the model is mathematically well-posed. If it is, then those using the model can be more confident in its numerical results. If it is not, the current formulation may have to be abandoned or modified.

Section 2 presents a summary of the key formulations of the DCA model for ease of reference. In Section 3 the stability and well-posedness of the DCA model are examined through the analysis of small perturbations to a steady-state solution of a uniaxial stress problem. A detailed numerical example of analysis on silicon carbide (SiC) is presented in Section 4. Section 5 concludes and summarizes this work.

2. Summary of the DCA model

2.1. Damage tensor

In the DCA model the strain in the material is given by [5]

$$\boldsymbol{\varepsilon} = (\mathbf{C}_m + \mathbf{D}(\bar{c}))\boldsymbol{\sigma}, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the applied stress, $\mathbf{C_m}$ is the compliance (4th-order tensor) of the matrix (in the current work, $\mathbf{C_m}$ is taken as isotropic and is specified by the bulk and shear moduli). The damage tensor $\mathbf{D}(\bar{c})$ is given by [5]

$$\mathbf{D}(\bar{c}) = \beta^e N_0 \bar{c}^3 \left(\frac{3}{2-\nu}\mathbf{P}^d + \mathbf{P}^+ \left(\mathbf{P}^d + \frac{5}{2}\mathbf{P}^{sp} \right) \mathbf{P}^+ \right), \tag{2.2}$$

where $\beta^e \equiv 64\pi(1-\nu)/(15G)$ is a material constant; ν and G are, respectively, the Poisson's ratio and shear modulus of the matrix. N_0 is the crack number density per solid angle and $\bar{c}(t)$ is the mean crack radius [5]. In the model N_0 is kept as a material constant, and the damage in the material is reflected through the evolution of $\bar{c}(t)$. $\mathbf{P}^d$, $\mathbf{P}^{sp}$ and $\mathbf{P}^+$ are, respectively, the spherical, deviatoric and positive projection operators [5].

2.2. Damage evolution

The evolution of the mean crack size $\bar{c}(t)$ is given by [5]

$$\frac{\dot{\bar{c}}}{\dot{c}_{\max}} = 1 - \frac{1}{1 + \langle F(\boldsymbol{\sigma}, \bar{c}) \rangle}, \tag{2.3}$$

where $F(\boldsymbol{\sigma}, \bar{c})$ is the damage function and the angled bracket is the Macaulay bracket and the maximum growth rate $\dot{c}_{\max}$ is the terminal speed for crack growth (see [5] for details).

The expressions for $F(\boldsymbol{\sigma}, \bar{c})$ under a general stress state were given previously [5]. In the current work, we focus on uniaxial-stress tension where the damage surface reduces to that of the Rankine maximum tensile criterion for brittle materials:

$$\begin{aligned} F(\boldsymbol{\sigma}, \bar{c}) &= \frac{\sigma_1}{\sigma_{cr}(\bar{c})} - 1 \\ \sigma_{cr}(\bar{c}) &\equiv \sqrt{\frac{\pi}{1-\nu}\frac{G\gamma}{\bar{c}}} \end{aligned} \tag{2.4}$$

where $\sigma_1 > 0$ is the uniaxial stress in the material, γ is the effective surface energy of the material, and $\sigma_{cr}(\bar{c})$ is the tensile strength of the material, which decreases with the mean crack size $\bar{c}$.