Applied Mathematical Modelling 40 (2016) 3832–3842

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Some new results about the variance inactivity time ordering with applications

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ARTICLE INFO

Article history:
Received 1 December 2013
Revised 20 June 2015
Accepted 2 November 2015
Available online 10 November 2015

Keywords:
Mean inactivity time
Reversed hazard rate
Preservation
Mixture
Random sum
Shock models

ABSTRACT

The purpose of this paper is to enhance the study of the variance inactivity time order. We first present some new implications and characterizations concerning this order. Then, we develop some preservation properties of this order under some reliability operations such as mixture, and convolution. Finally, we highlight some new applications of this order in the context of statistics and reliability theory.

1. Introduction and motivation

One of the main objectives of statistics is the comparison of random quantities. These comparisons are mainly based on the comparison of some measures associated with these random quantities. The need for providing a more detailed comparison of two random quantities has been the origin of the theory of stochastic orders that has grown significantly during the last 40 years. We refer the readers to [1] for an exhaustive monograph on this topic.

Let X and Y be two lifetime random variables having distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, and density functions f and g, respectively. For the lifetime random variables X and Y, denote $X_{(t)} = [t - X \mid X \le t]$ and $Y_{(t)} = [t - Y \mid Y \le t]$, for fixed $t > 0$, which have distribution functions $F_{(t)}(s) = P[t - X \le s \mid X \le t]$ and $G_{(t)}(s) = P[t - Y \le s \mid Y \le t]$, respectively. These conditional random variables are known in the literature as inactivity times (cf. [2–4]). Throughout the paper we will use the term increasing in place of non-decreasing, and decreasing in place of non-increasing. The simplest way to compare the distribution functions of inactivity times is through comparison of their associated means (cf. [5–15]). For all $t \ge 0$, let $m_X(t) = E(X_{(t)})$ and $m_Y(t) = E(Y_{(t)})$ denote the mean inactivity time functions of X and Y, respectively. Given two lifetime random variables X and Y, we say that X is mean inactivity time-smaller than Y (denoted as $X \le_{MIT} Y$) if, and only if $m_X(t) \ge m_Y(t)$,

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http://dx.doi.org/10.1016/j.apm.2015.11.006

for all $t > 0$, or equivalently if

$$\frac{\int_0^t G(u)du}{\int_0^t F(u)du} \text{ is increasing in } t > 0.$$

However, the means sometimes do not exist and therefore they are often not very informative. In many instances in applications one has more detailed information, for the purpose of comparison of two inactivity time distribution functions, than just the two means. When one wishes to compare two inactivity time distribution functions that have non-ordered means, one is usually interested in the comparison of the dispersion of these distributions. For example, suppose that the random variable X_i represents the failure time of the ith component of the system for each $i = 1, 2, \ldots, n$. Assume that the component with lifetime X_i, fails at time t or sometimes before than t and consequently the system can be considered as a black box in the sense that the exact failure time of X_i in it is unknown. It is of interest to estimate the times that has elapsed since the ith components failure and to study the dispersion of this elapsed interval of time. As a result, some authors have considered properties of stochastic orders which are defined on the basis of the variance inactivity time function of $\sigma_X^2(t) = Var(t - X \mid X \leq t)$ (cf. [16–18]).

Given two lifetime random variables X and Y, we say that X is smaller than Y in variance inactivity time (denoted as $X \leq_{VIT} Y$) if, and only if

$$\frac{\int_0^t \int_0^x G(u)\,dudx}{\int_0^t \int_0^x F(u)\,dudx} \text{ is increasing in } t > 0. \tag{1}$$

The purpose of this paper is to provide new characterizations, preservation properties and applications for the variance inactivity time (VIT) order. The organization of this paper is as follows. In Section 2, we provide some implications and characterizations regarding the VIT order. In Section 3, several preservation properties of the VIT order under reliability operations of mixture and convolution are discussed. Some applications in the context of statistics and reliability are provided in Section 4. Finally, in Section 5, we give a brief conclusion, and some remarks of the current and future of this research.

2. Basic properties, characterizations and implications

In this section, we focus on some preliminary properties of the VIT order including some characterizations and implications. First, for the lifetime random variable X, let us propose the following measure. For all $t > 0$, define

$$\nu_X(t) = \left[\frac{d}{dt}\ln\int_0^t\int_0^x F(u)\,dudx\right]^{-1}$$

$$= \frac{\int_0^t \int_0^x F(u)\,dudx}{\int_0^t F(x)\,dx}.$$

Similarly, define ν_Y for the lifetime random variable Y. It is noticeable that this measure can be written in terms of moments ratio of the inactivity time as

$$\nu_X(t) = \frac{E((t-X)^2 \mid X \leq t)}{2E(t - X \mid X \leq t)}, \text{ for all } t > 0.$$

We observe that this measure is useful to establish the VIT order between two random variables. The condition given in the following definition for the VIT order is equivalent to (1).

Definition 1. The lifetime random variable X is said to be smaller than Y in variance inactivity time (denoted as $X \leq_{VIT} Y$) if, and only if

$$\nu_X(t) \geq \nu_Y(t), \quad \text{for all } t > 0.$$

Based on the VIT function, a new class of lifetime distributions is introduced and studied (cf. [5,16] and [17]).

Definition 2. The lifetime random variable X is said to have an increasing variance inactivity time (IVIT) if $\sigma_X^2(t)$ is increasing in $t > 0$, or equivalently if $\nu_X(t)$ is increasing in $t > 0$.

The next result provides a useful characterization of the VIT order.

Theorem 1. *Given two lifetime random variables X and Y, $X \leq_{VIT} Y$ holds if, and only if*

$$\frac{\int_0^t xF(x)\,dx}{\int_0^t F(x)\,dx} \leq \frac{\int_0^t xG(x)\,dx}{\int_0^t G(x)\,dx}, \quad \text{for all } t > 0.$$