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Stress and strain mapping tensors and general work-conjugacy in large strain continuum mechanics

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A B S T R A C T

In this paper we show that mapping tensors may be constructed to transform any arbitrary strain measure in any other strain measure. We present the mapping tensors for many usual strain measures in the Seth–Hill family and also for general, user-defined ones. These mapping tensors may also be used to transform their work-conjugate stress measures. These transformations are merely geometric transformations obtained from the deformation gradient and, hence, are valid regardless of any constitutive equation employed for the solid. Then, advantage of this fact may be taken in order to simplify the form of constitutive equations and their numerical implementation and thereafter, perform the proper geometric mappings to convert the results –stresses, strains and constitutive tangents– to usually employed measures and to user-selectable ones for input and output. We herein provide the necessary transformations. Examples are the transformation of small strains formulations and algorithms to large deformations using logarithmic strains.

1. Introduction

Whereas in small strain continuum mechanics there is no debate about which ones are the stress and strain measures to be used in constitutive equations, at large strains the options are multiple. Regarding large strains, the Seth–Hill [1,2] family of strain measures (see also the previous work [3]) are typically used, although some other deformation measures are being proposed [4]. Different authors have different preferences over the strain measures. For example, in large strain hyperelasticity it is typical to use the Cauchy–Green deformation tensor (see for example [5–7]), or alternatively the Green–Lagrange strain tensor. Deformation invariants used in anisotropic hyperelasticity are almost always defined from the Cauchy–Green deformation tensor [5]. The reason for this choice is that the Cauchy–Green deformation tensor and the Green–Lagrange strain tensor are directly obtained from the deformation gradient and the latter from the gradient of the displacements. Hence, they are naturally included in the Updated Lagrangian and Total Lagrangian formulations in finite element codes [8,9]. Logarithmic strains are also a good choice not only for hyperelasticity [10–12] and visco-hyperelasticity [13–15], but specially for plasticity [16–22]. It has been shown that a linear relation between logarithmic strains and Kirchhoff stresses yield a rather accurate prediction of the behavior of some metals and polymers [23,24]. Furthermore, the use of a quadratic hyperelastic energy function of the logarithmic strains and an exponential integration allows for simple, yet accurate stress integration algorithms in large strain elasto-plasticity, where a small

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strain integration is employed teamed with geometric pre- and postprocessors [17,21,22]. Logarithmic strains have arguably also a more intuitive and meaningful interpretation, not only for uniaxial loading but also for shear terms [25,26].

However, one of the issues usually not well treated in the literature and, hence, which yields some misunderstandings is the fact that the choice of one strain measure over another is essentially a matter of tradition and can be also a matter of convenience. Furthermore, stresses and strains for user input and output should be selectable by the user, independently of the material model being employed. One of the purposes of this paper is to show that any strain measure may be directly related to any other strain measure and then, the proper work-conjugate stress measure must be employed, which remarkably transforms using equivalent relations. Furthermore, generalized strain measures, not only the Seth–Hill bundle [1,2], may be used if they are more convenient for the purpose, for example in order to possibly establish linear constitutive relations between stresses and strains as, for example in [16–22] and in [4] in a more general context. Then, the transformation from any strain measure (for example the deformation gradient or the Green–Lagrange strain) to the generalized one is simply performed using the proper mapping tensor which we also introduce. In a similar way, the transformation of the resulting generalized stress measure to Cauchy or Piola stresses, or the resulting constitutive tangent, may also be performed using similar mapping tensors. An important point is that these transformations are valid regardless of the constitutive equations for the material and of the material symmetries. In fact, we remark that no constitutive equation will be used throughout the paper except in the examples. In essence, they can be considered as deformation measures in locally transformed bodies. Invariants for constitutive equations may also be defined using these generalized strain measures.

In the following section of the paper we depart from the stress power to establish power conjugacy from scratch. Then we introduce the stress and strain mapping tensors for most of the typically used strain and their work-conjugate stress measures. Finally we introduce generalized strain measures, their work-conjugate stress measures and the mapping between two arbitrary sets. We further derive the transformations for general constitutive equations from any stress/strain couple to any other one. We will assume a Cartesian representation to simplify the exposition, but of course the results are valid regardless the system of representation employed.

2. The stress power and work-conjugacy

Assume we have a body with an original volume ${}^0V$ and a deformed volume ${}^tV$, surrounded respectively by ${}^0S$ and ${}^tS$. A point representing an infinitesimal volume is denoted in the reference volume by ${}^0\boldsymbol{x}$, and in the current volume by

$$ {}^t\boldsymbol{x} = {}^0\boldsymbol{x} + {}^t\boldsymbol{u} \tag{1} $$

where ${}^t\boldsymbol{u}$ are the displacements. The body forces per unit current volume at time t are $\boldsymbol{b}$ and the surface ones (per unit current surface) are $\boldsymbol{t}$. Then by equilibrium of forces

$$ \int_{{}^tV} \boldsymbol{b}\, d\,{}^tV + \int_{{}^tS} \boldsymbol{t}\, d\,{}^tS = \boldsymbol{0} \tag{2} $$

By definition of the *Cauchy stress tensor* $\boldsymbol{\sigma}$—Cauchy's tetrahedron

$$ \boldsymbol{t}({}^t\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{\sigma}({}^t\boldsymbol{x}) \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{\sigma}({}^t\boldsymbol{x}) \tag{3} $$

where $\boldsymbol{n}$ is the unit vector normal to the plane related to the stress vector $\boldsymbol{t}$ and where the dot implies an index contraction, i.e. a scalar product in the case of vectors. The second identity holds because of equilibrium of angular moments. Then

$$ \int_{{}^tV} \boldsymbol{b}\, d\,{}^tV + \int_{{}^tS} \boldsymbol{n} \cdot \boldsymbol{\sigma}\, d\,{}^tS = \boldsymbol{0} \tag{4} $$

and by the generalized Gauss theorem —see Eq. (5.1.5) of Ref. [27]

$$ \int_{{}^tV} (\boldsymbol{b} + \nabla \cdot \boldsymbol{\sigma}) d\,{}^tV = \boldsymbol{0} \tag{5} $$

where $\nabla \cdot \boldsymbol{\sigma}$ is the divergence of the Cauchy stress tensor respect to the current coordinates. By the localization theorem the well known local equilibrium equation is obtained —c.f. Eq. (5.3.5) of Ref. [27]

$$ \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0 \tag{6} $$

Aside, if $\boldsymbol{v}$ is the velocity field at time t, such that

$$ \boldsymbol{v} = {}^t\dot{\boldsymbol{x}} = {}^t\dot{\boldsymbol{u}} \tag{7} $$

the mechanical power is

$$ \mathcal{P} = \int_{{}^tV} \boldsymbol{b} \cdot \boldsymbol{v}\, d\,{}^tV + \int_{{}^tS} \boldsymbol{t} \cdot \boldsymbol{v}\, d\,{}^tS \tag{8} $$

Then using again Eq. (3) and the generalized Gauss theorem

$$ \mathcal{P} = \int_{{}^tV} \boldsymbol{b} \cdot \boldsymbol{v}\, d\,{}^tV + \int_{{}^tS} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{v}\, d\,{}^tS \tag{9} $$