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Interpolated variational iteration method for initial value problems

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ABSTRACT

In order to solve an initial value problem by the variational iteration method, a sequence of functions is produced that converges to the solution under suitable conditions. In the non-linear case, the terms of the sequence become complicated after a few iterations, and thus computing a highly accurate solution is difficult or even impossible. In this study, we propose a new approach for one-dimensional initial value problems, which is based on approximating each term of the sequence by a piecewise linear function. Moreover, we prove the convergence of the method. Three illustrative examples are given to demonstrate the superior performance of the proposed method compared with the classical variational iteration method.

1. Introduction

He's variational iteration method (VIM) [1,2] is a powerful mathematical technique for solving linear and nonlinear problems, which is easy to implement in practice. It has been employed successfully for solving various ordinary differential equations (ODEs) and partial differential equations [1–6]. The convergence of the VIM has also been investigated in previous studies [7,8]. Zhao and Xiao [9] used the VIM to solve singular perturbation initial value problems and investigated its convergence. Recently, Wu and Baleanu [10] introduced a new method for defining the Lagrange multipliers in the VIM to solve fractional differential equations with the Caputo derivatives. They also developed the VIM for q-fractional difference equations in [11]. A review article [12] considered some new applications of the VIM to numerical simulations of differential equations and fractional differential equations.

Consider the following differential equation:

$$\mathcal{L}u(t) + \mathcal{N}u(t) = g(t), \tag{1}$$

where $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively, and g is an inhomogeneous term. Given an initial estimate, $u_0(t)$, the VIM required to solve (1) takes the following form:

$$u_{m+1}(t) = u_m(t) + \int_{t_0}^{t} \lambda(s,t)(\mathcal{L}u_n(s) + \mathcal{N}u_m(s) - g(s))ds, \quad m = 0, 1, 2, \ldots, \tag{2}$$

where λ is a general Lagrange multiplier (for more details, see [1,2,4]).

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Clearly, the VIM generates a sequence of functions that can converge to the solution of the problem under suitable conditions. After a few iterations, each term of this sequence often involves a definite integral, where the integrand contains several nonlinear terms. In this case, computing a high accuracy solution is difficult or even impossible using standard software such as MAPLE, MATHEMATICA, or MATLAB. Abbasbandy [13] described an application of the VIM to quadratic Riccati differential equations, which illustrates this issue. To overcome this problem, Geng et al. [14] introduced a piecewise VIM for the quadratic Riccati differential equation, where the main idea is to solve the problem over a large interval, so the interval is split into several subintervals and the problem is solved for each subinterval by the VIM in a progressive manner. Thus, the accuracy of the computed solution can be improved considerably, but the main problem is still unresolved because the VIM is implemented directly for each subinterval.

In this study, to solve one-dimensional initial value problems, we modify the VIM so each term of the sequence is interpolated by a piecewise linear function. In the following, we refer to the proposed method as interpolated VIM (IVIM). Unlike the VIM, the IVIM does not require any symbolic computation and all of the computations are performed numerically. Therefore, we can compute hundreds of sequence terms within a small amount of time.

The remaining of the paper is organized as follows. We introduce the IVIM in Section 2. The convergence of the proposed method is demonstrated in Section 3. In Section 4, we present three numerical examples. We give some concluding remarks in Section 5.

2. The IVIM

Consider the one-dimensional initial value problem:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [a, T], \\ u(a) = u_a. \end{cases} \tag{3}$$

For simplicity, we assume that $u_a = 0$; otherwise, we can use a simple change of variable $\tilde{u} = u - u_a$ to obtain $\tilde{u}(a) = 0$. In this case, Eq. (2) becomes:

$$u_{m+1}(t) = u_m(t) + \int_a^t \lambda(s, t)(u_m'(s) - f(s, u_m(s)))ds, \quad m = 0, 1, 2, \ldots, \tag{4}$$

where $u_0(t)$ satisfies the initial condition $u_0(a) = 0$. After integrating by parts, Eq. (4) can be rewritten as:

$$u_{m+1}(t) = G_m(t) - \int_a^t H_m(s, t)ds, \tag{5}$$

where,

$$G_m(t) = (1 + \lambda(t, t))u_m(t) - \lambda(a, t)u_m(a),$$

and

$$H_m(s, t) = \frac{\partial \lambda}{\partial s}(s, t)\, u_m(s) + \lambda(s, t) f(s, u_m(s)).$$

In order to present the IVIM, we take a natural number n and discretize the interval $[a, T]$ into $n - 1$ subintervals with a step size of $h = (T - a)/(n - 1)$ and grid points:

$$t_i = a + (i - 1)h, \quad i = 1, 2, \ldots, n.$$

Now, we define the well-known B-spline basis functions (see [15]) of first order on the nodal points t_i, i.e.,

$$\varphi_i(t) = \begin{cases} \dfrac{t - t_{i-1}}{h}, & t_{i-1} \le t < t_i, \\ \dfrac{t_{i+1} - t}{h}, & t_i \le t \le t_{i+1}, \quad i = 2, 3, \ldots, n-1, \\ 0, & t < t_{i-1} \text{ or } t > t_{i+1}, \end{cases}$$

$$\varphi_n(t) = \begin{cases} \dfrac{t - t_{n-1}}{h}, & t_{n-1} \le t \le t_n, \\ 0, & t < t_{n-1}, \end{cases}$$

on $[a, T]$. Each function φ_i is equal to 1 at the grid point t_i but equal to zero at the other grid points. Let,

$$X_h = \text{span}\{\varphi_i : i = 2, 3, \ldots, n\}.$$

Every $v^{(h)}$ in X_h is a piecewise linear function of the form:

$$v^{(h)}(t) = \sum_{i=2}^{n} \alpha_i \varphi_i(t), \quad t \in [a, T].$$