Applied Mathematical Modelling 40 (2016) 3991–4001

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Solving differential-algebraic equations through variational iteration method with an auxiliary parameter

H. Ghaneai [a,b,*], M.M. Hosseini [b,c]

[a] Young Researchers and Elite Club, Islamic Azad University, Ashkezar Branch, Ashkezar, P.O. Box 8941613695, Iran
[b] Faculty of Mathematics, Yazd University, P.O. Box 89195-741, Yazd, Iran
[c] Department of Applied Mathematics, Faculty of mathematics and Computer, Shahid Bahonar University of Kerman, P.O. Box 76169-14111, Kerman, Iran

ARTICLE INFO

Article history:
Received 2 December 2013
Revised 9 May 2015
Accepted 5 October 2015
Available online 10 November 2015

Keywords:
Auxiliary parameter
Differential-algebraic equations
Variational iteration method

ABSTRACT

In this study, we consider the importance of an auxiliary parameter, which is introduced into the well-known variational iteration method to obtain solutions for differential-algebraic equations. We study the convergence of the proposed method, which is called the variational iteration method with an auxiliary parameter. In addition, the proposed method is applied to two differential-algebraic equations to elucidate the solution procedure and to select the optimal auxiliary parameter. Comparisons with the results obtained by the standard variational iteration method demonstrate that the auxiliary parameter is highly effective in controlling the convergence region of the approximate solution.

1. Introduction

Differential-algebraic equations (DAEs) are systems of differential equations (which are sometimes referred to as descriptors, singular or semi-state systems), where the unknown functions satisfy additional algebraic equations [1]. In other words, they comprise a set of differential equations with additional algebraic constraints. Many physical problems are governed by a system of DAEs, e.g., for electrical networks, constrained mechanical systems of rigid bodies, singular instance perturbation and discretization of partial differential equations, and control theory [2,3]. In recent years, the problem of finding the solutions to these equations has been addressed by many investigators. In this study, we consider a linear variable coefficients system of DAEs:

$$\begin{cases} \mathbf{A}(t)\dot{Y} + \mathbf{B}(t)Y = \mathbf{f}(t), & \\ Y(0) = y_0, & t \in [0, T]. \end{cases} \tag{1}$$

where $\mathbf{A}(t)$ is a singular matrix, $\mathbf{B}(t)$ is a nonsingular matrix, and $\mathbf{f}(t) : [0, T] \longrightarrow \mathbb{C}^m$ is a supposed continuous function. Implicit Runge–Kutta and the backward differentiation formula methods are the most important numerical methods for certain classes of DAEs [4]. However, these methods cannot be employed to approximate the solutions of all DAEs [5]. In some cases, a DAE can be converted into a system of ordinary differential equations (ODEs), but the numerical stability of the system is often undermined during this process, so even if all of the DAEs can be converted into ODEs, it is usually undesirable to do so [6–8]. Many studies have investigated numerical methods for solving DAEs. Ascher, Petzold, Campbell, and Gear performed extensive studies of the numerical solutions for this class of equations [9–11]. The related predicted sequential regularization method and sequential

* Corresponding author. Tel.: +98 3531232715.
E-mail addresses: h.ghaneai@gmail.com, h.ghaneai@stu.yazd.ac.ir (H. Ghaneai), hosse_m@yazd.ac.ir (M.M. Hosseini).

http://dx.doi.org/10.1016/j.apm.2015.10.002

regularization method are numerical methods, which were designed to deal with certain classes of DAEs [12,13]. Waveform relaxation methods were proposed to solve the initial value problems for DAEs and they have been investigated by many authors [14–16]. The approximate analytical solutions to DAEs were obtained by Hosseini using the Adomian decomposition method [17]. Moreover, the homotopy perturbation method can be used to obtain approximate solutions of DAEs [18].

The variational iteration method (VIM) was proposed and developed further by He [19–22]. This method can be employed to solve various types of functional equations. In fact, a large class of nonlinear problems converges rapidly to approximate solutions using this method. Moreover, various nonlinear equations can be solved with this method [23–25]. Yilmaz and Inc proposed a VIM with an auxiliary parameter to control the convergence rate, but they did not provide a general rule for selecting the best auxiliary parameter [25]. This improved method was developed further by Hosseini et al. who introduced some useful rules to facilitate the determination of the optimal auxiliary parameter [23,26–28].

In the present study, we propose a new VIM with an auxiliary parameter to obtain the approximate solution of DAEs. In the proposed method, the residual function and the error of the residual function are defined to select the optimal auxiliary parameter. In addition, we study the convergence of VIM with an auxiliary parameter according to an alternative approach using this method.

2. VIM with an auxiliary parameter

In this section, we describe the VIM with an auxiliary parameter. Consider the following general nonlinear equation:

$$Hu = Lu + Nu + Ru + g(x,t) = 0, \tag{2}$$

where L denotes the highest order derivative, which is assumed to be easily invertible, R indicates a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source inhomogeneous term. He proposed VIM with a correction functional for (2), which can be written as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau) H u_n(x,\tau)d\tau. \tag{3}$$

In the equation above, λ is a Lagrange multiplier, which can be identified optimally using the variational theory [20], u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\partial\tilde{u}_n = 0$. The approximations $u_{n+1}(x,t),\ n \geq 0$, of the solution $u(x, t)$, are readily obtained using the determined Lagrangian multiplier and any selected function $u_0(x, t)$, provided that $Lu_0(x,t) = 0$. In fact, the correction Functional (3) will yield several approximations such as the following,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{4}$$

In summary, we give the following variational iteration formula for (2):

$$\begin{cases} u_0(x,t) \text{ is an arbitrary function}, \\ u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau) H u_n(x,\tau)d\tau, \qquad n \geq 0. \end{cases} \tag{5}$$

An unknown auxiliary parameter can be inserted into the variational iteration algorithm, Eq. (3):

$$\begin{cases} u_0(x,t) \text{ is an arbitrary function}, \\ u_1(x,t,h) = u_0(x,t) + h\int_0^t \lambda(\tau) H u_n(x,\tau)d\tau, \\ u_{n+1}(x,t,h) = u_n(x,t,h) + h\int_0^t \lambda(\tau) H u_n(x,\tau,h)d\tau, \qquad n \geq 1. \end{cases} \tag{6}$$

The successive approximate solutions $u_{n+1}(x,t,h)$, $n \geq 1$ include the auxiliary parameter h. The accuracy of this method depends on the assumption that the approximation $u_{n+1}(x,t,h),\ n \geq 1$ converges to the exact solution. In the proposed method, the auxiliary parameter ensures that this assumption is satisfied. In general, it is straightforward to select an appropriate value of h to ensure that the approximate solutions are convergent, which is achieved based on the error of norm two of the residual function [26–29]. In fact, the suggested approach of VIM with an auxiliary parameter [30] is very simple, easier to apply, and capable of approximating the solution more precisely in a large solution domain.

3. Convergence analysis

In this section, we consider the convergence of the VIM with an auxiliary parameter according to an alternative approach using this method, which we describe in the following. This approach can be implemented in a reliable and efficient manner, and it can also handle nonlinear differential Eq. (2). The linear operator L is defined as $L = \frac{d}{dt} + \alpha$ when the VIM with an auxiliary parameter is employed to solve the DAE (1). Now, the operator A is defined as follows:

$$Au(t,h) = h\int_0^t \lambda(\tau) H u(\tau,h)d\tau, \tag{7}$$