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Behavior of limit cycle bifurcations for a class of quartic Kolmogorov models in a symmetrical vector field

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ABSTRACT

In this study, we consider the limit cycle bifurcation problem for a class of quartic Kolmogorov models with five positive singular points, i.e., (1,1), (1,2), (2,1), (1,3), and (3,1), which lie in a symmetrical vector field relative to the line $y = x$. We classify these singular points. We show that points (1,2) and (2,1) can bifurcate into three small limit cycles by simultaneous Hopf bifurcation, and that points (1,3) and (3,1) can bifurcate into three small limit cycles by simultaneous Hopf bifurcation. In addition, we construct limit cycles for this model and we show that four positive singular points, i.e., (1,1), (1,2), (2,1), and (1,3), can bifurcate into eight limit cycles in total, among which six cycles may be stable. Few previous studies have considered a symmetrical Kolmogorov model with several positive singular points. Our results are good in terms of the Hilbert number for the Kolmogorov model.

1. Introduction

Kolmogorov models comprise a significant class of ecological models that are used widely in ecology to represent the dynamic behavior of prey and predators, which are expressed in the following form:

$$\frac{dx}{dt} = x\,f(x,y), \quad \frac{dy}{dt} = y\,g(x,y), \tag{1}$$

where $f(x, y)$ and $g(x, y)$ are polynomials about x, y and x, y that represent the density of the predator and prey, respectively. In this case, we only consider the behavior of the orbits in the "realistic quadrant" $\{(x, y): x > 0, y > 0\}$. The existence of limit cycles and the number of limit cycles that can arise from positive equilibrium points are of particular significance in applications because a limit cycle corresponds to an equilibrium state of the system. The existence and stability of limit cycles are related to the positive equilibrium points. In addition, the problem of the number of limit cycles is closely related to Hilbert's 16th problem, which is an important unsolved problem. The second part of Hilbert's 16th problem was posed in 1902 where the aim is to find the Hilbert number, H(m), which denotes the maximal number of limit cycles of polynomial systems of degree m. The Hilbert number has been studied widely and many good results regarding limit cycle bifurcations have been reported (see [1,2]). Thus, many studies of Kolmogorov models have considered the limit cycle problem. For example, previous studies [3,4] considered a class of cubic Kolmogorov systems with three limit cycles. Du et al. [5] reported a class of cubic Kolmogorov system that could

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bifurcate into five limit cycles, including three stable cycles. Du and Huang [6] described a cubic Kolmogorov system with five limit cycles, among which four cycles are stable. Lloyd et al. [7] considered a cubic Kolmogorov system that could bifurcate into six limit cycles. Several studies [8–10] have investigated a general Kolmogorov model and obtained the conditions for the existence and uniqueness of limit cycles, as well as classifying a series of models. Du et al. [11] investigated the bifurcation of limit cycles for a class of quartic Kolmogorov models with two symmetrical positive singular points. Du et al. [12] also studied the three-dimensional Hopf bifurcation for a class of cubic Kolmogorov models. Many good results have been obtained in terms of the limit cycles of Kolmogorov models, especially for lower degree systems by only analyzing the state of positive equilibrium points. However, few results have been obtained for simultaneous limit cycles that bifurcate from several different equilibrium points, possibly because it is difficult to investigate this type of problem. From an ecological perspective, investigating multiple positive equilibrium points is meaningful. The equilibrium point (a, b) indicates that the density of the predator relative to the prey is a: b and it is possible for several equilibrium points to occur in an ecosystem, e.g., the density of the predator relative to the prey is 1: 2 if the predator weighs x kg, and the density of the predator relative to the prey is 1: 3 if the predator weighs $1.5x$ kg, but system (1) only describes the density ratio without considering other conditions related to the predator.

In this study, we consider a class of quartic Kolmogorov models, as follows:

$$\begin{cases} \frac{dx}{dt} = x(y-1)[a + A_{01}x + A_{10}y + A_{11}xy - \frac{1}{5}(A_{10}+A_{11})y^2 \\ \quad -\frac{1}{5}(5a + 5A_{01} + 6A_{10} + 6A_{11})x^2] = P(x,y), \\ \frac{dy}{dt} = y(x-1)[a + A_{10}x + A_{01}y + A_{11}xy - \frac{1}{5}(A_{10}+A_{11})x^2 \\ \quad -\frac{1}{5}(5a + 5A_{01} + 6A_{10} + 6A_{11})y^2] = Q(x,y), \end{cases} \tag{2}$$

where $a, A_{10}, A_{11}, A_{01} \in \mathbf{R}$.

Clearly, model (2) has five positive equilibrium points, i.e., (1,1), (1,2), (2,1), (1,3), and (3,1). In fact, the vector field of system (2) is symmetrical relative to the line $y = x$. We focus on the limit cycle bifurcation of these positive equilibrium points. We employ a previously described method [13–15] to compute the focal values and we study the limit cycle bifurcation. To investigate the limit cycle bifurcation problem in a planar vector field, the method described in [13–15] is valid and used widely, e.g., see our results in [16–19]. By computing the calculations carefully, we obtain expressions of the first three focal values for each positive equilibrium point. We show that each of the two positive equilibrium points, i.e., (1,2) and (2,1), can bifurcate into three limit cycles by simultaneous Hopf bifurcation and that each of the two positive equilibrium points, i.e., (1,3) and (3,1), can bifurcate into three limit cycles by simultaneous Hopf bifurcation. In addition, we provide the distribution structure of the limit cycles for model (2) and we show that four positive singular points can bifurcate into eight limit cycles, among which six cycles can be stable. It should be noted that we obtain interesting bifurcation behavior in terms of the the limit cycle bifurcations of a Kolmogorov model with multiple positive equilibrium points in a symmetrical vector field. Obviously, the results obtained for the symmetric system (1) are interesting in terms of the theoretical analysis.

The remainder of this paper is organized as follows. In Section 2, we introduce the method for studying limit cycle bifurcations [13–15], as well as giving the recursive formulae for the singular point values and the method for finding the Hilbert number. In Section 3, we discuss the quality of the five positive equilibrium points and their classifications. In Section 4, we obtain the focal values of the positive equilibrium points, i.e., (1,2) and (1,3), of model (2) based on careful computations and simplification, and we show that each of the positive equilibrium points of model (2), i.e., (1,2) and (1,3), has three focal values. Moreover, we discuss the bifurcation of the limit cycles for model (2) and we show that each of the two positive equilibrium points of model (2), i.e., (1,2) and (1,3), can bifurcate into three small limit cycles. In Section 5, we provide the distribution structure on limit cycles for model (2) and we show that model (2) can yield eight small simultaneous limit cycles from four positive equilibrium points, among which six limit cycles may be stable. In Section 6, we give our conclusions and some discussion. Our results are good in terms of the Hilbert number or the number of stable limit cycles for the Kolmogorov model.

2. Method for studying limit cycle bifurcations

In order to study limit cycle bifurcations, it is often important to compute the focal values or Lyapunov constants. We use an algorithm based on the singular point values to compute the focal values (or Lyapunov constants) and to study the limit cycle problem. Previous studies [13–15] described the relationship between the focal values and singular point values, and we introduce these results in the following.

Consider the following real system

$$\begin{cases} \frac{dx}{dt} = \delta x - y + \sum_{k=2}^{\infty} X_k(x,y), \\ \frac{dy}{dt} = x + \delta y + \sum_{k=2}^{\infty} Y_k(x,y), \end{cases} \tag{3}$$

where $X_k(x,y) = \sum_{\alpha+\beta=k} A_{\alpha\beta}x^{\alpha}y^{\beta}$, $Y_k(x,y) = \sum_{\alpha+\beta=k} B_{\alpha\beta}x^{\alpha}y^{\beta}$,