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# Negative ageing properties for counting processes arising in virtual age models

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ABSTRACT

This paper is focused on counting processes of interest in reliability theory. In particular, we study virtual age models. Our aim is to consider these counting models stopped at a random time $T$, independent of the process. We investigate the negative ageing properties for the stopped process (decreasing failure rate and log-convexity), provided that $T$ has the analogous properties. For instance, we show that these properties hold for the so-called Kijima type I virtual age models, of interest in reliability, under quite general assumptions.



## 1. Introduction

Counting processes arise in many areas of applied probability, such as queueing theory, insurance mathematics, and reliability, to mention a few. A counting process is a stochastic process $\{N(t): t \geq 0\}$ in which $N(t)$ counts the number of events up to a fixed time $t$. Given a counting process, an interesting problem is to count the number of events in a random interval $T$. For instance, if we are concerned with the number of repairs of a piece in a machine, $N(T)$ can represent the number of repairs up to the operational time $T$ of this machine (see [1, p. 42]). We can also find examples of the application of the random variable $N(T)$ in queueing models [2], in order to evaluate the stationary number of customers waiting in a queue, in wireless communication systems [3], and in the health sciences [4]. See also [5–7] for more applications.

In reliability and other areas of applied probability, different probabilistic concepts of ageing are of interest, defining different classes of life distribution, e.g., the increasing failure rate (IFR) and the decreasing failure rate (DFR) distributions. See [8], for instance, for a thorough study of this issue. In particular, much research has been done in the study of the ageing properties of $N(T)$ in terms of the ageing properties of $T$, especially when $T$ is independent of the counting process. The particular case of the Poisson process has been widely studied (see [9, ch. 7] and the references therein). This has been generalized to renewal processes in [10]. When the inter-arrival periods are dependent, the IFR property has been analyzed in [11] and in [12]. Recently, Badía and Cha [13] have considered models in which the random interval $T$ is dependent on the counting process.

In this paper we will study the ageing properties of $N(T)$ when the counting process has dependent interarrival times. This study will include virtual age models (see [14, Ch. 2.6] and the references therein). They were introduced by Kijima and Sumita [15] in order to generalize renewal theory by considering processes exhibiting a Markovian dependence between the

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arrival times. Later, they were shown to be of interest in reliability theory, in order to model the repairs of machines in which the degree of repair can be imperfect [16].

We will focus our attention on the preservation of negative ageing properties. These properties describe the lifetime of systems which improve with age. In particular, we will study the DFR and log-convexity properties, which will be defined in Section 2. As an example, the lifetime of a system whose distribution is a mixture of exponential random variables has these properties (see Proschan [17]).

It should be pointed out that usually the preservation of negative ageing properties under randomly stopped counting processes holds in more general situations than their corresponding positive ageing ones (discrete IFR or log-concavity). See, for instance [10], to compare the conditions under which the IFR and DFR properties are preserved in the context of independent inter-arrival times. It should also be pointed out that results concerning the preservation of the increasing failure rate for the specific case of virtual age models appear in [12]. The results concerning the preservation of the decreasing failure rate presented in this paper will hold for a more general class of virtual age models.

This paper is organized as follows. In Section 2, we will give some preliminary definitions used throughout this paper. In Section 4 we will demonstrate the DFR property for general counting processes exhibiting a strong decreasing condition between the inter-arrival times (Theorem 4.3). As an application we will demonstrate the DFR property for general virtual age models (Theorem 4.5) and, in particular, for the ones proposed in [15,16] (Corollary 4.6), under quite general assumptions. In Section 5 we give results concerning the preservation of log-convexity (which is a stronger concept than DFR) for the so-called Kijima type I models. As far as we know, very few results concerning the preservation of discrete log-convexity for counting processes are known in the literature. We can mention the well-known result for the Poisson process (see [9]) and its extension to a renewal process obtained in [10]. In Section 6 we give two simple applications of Theorem 4.5 and Corollary 5.2, concerning the DFR and log-convexity properties in frailty models (see Corollaries 6.1 and 6.2). We also consider a model in which the inter-arrival times and the stopping random variable can be dependent.

## 2. Preliminaries

Suppose $\{N(t)\colon t \geq 0\}$ is a general counting process. Denote its arrival times by $S_n$, $n = 0, 1, \dots$ $(S_0 = 0)$ and its inter-arrival epochs by $X_n$, $n = 1, 2, \dots,$ that is:

$$S_n = \sum_{i=1}^{n} X_i, \quad n = 1, 2, \dots$$

The only assumption about the $X_i$ is that they are non-negative random variables. Recall that the counting process (the number of arrivals up to instant $t$) is defined through the arrival epochs by the following expression

$$N(t) = \max\{n : S_n \leq t\}, \quad t \geq 0.$$

As mentioned in Section 1, we will study negative ageing properties for $N(T)$, in which $T$ is a random time. First, we give the definition of the properties involved in the concept of discrete log-convexity. Recall that a sequence $(a_n)_{n=0,1,\dots}$ is *log-convex* if $a_{n+1}^2 \leq a_n a_{n+2}$, $n = 0, 1, 2, \dots$.

**Definition 2.1.** Let $X$ be a non-negative integer-valued random variable.

(a) $X$ is said to be *discrete decreasing failure rate* (discrete DFR) if the sequence $\{P(X \geq n)\}_{n=0,1,\dots,}$ is log-convex.
(b) $X$ is said to be *discrete log-convex* if the sequence $\{P(X = n)\}_{n=0,1,\dots}$ is log-convex.

It is well known (see, for instance, [18, p. 257]) that discrete log-convexity implies discrete DFR.

Now, we define the negative ageing properties for a general random variable that involve the concept of log-convexity:

**Definition 2.2.** Let $I \subseteq \mathbb{R}^m$ be an interval. A function $f$: $I \to [0, \infty]$ is said to be log-convex (concave) on $I$ if for all $\mathbf{x}, \mathbf{y} \in I$ and $0 \leq \alpha \leq 1$ one has $f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq (\geq) f(\mathbf{x})^{\alpha} f(\mathbf{y})^{1-\alpha}$, or, equivalently, $\log f$ is convex (concave).

**Remark 2.3.** The following property will be extensively used throughout this paper. If $f$: $[0, \infty) \to [0, \infty)$ is log-convex, then

$$f(x)f(x+y+z) \geq f(x+y)f(x+z), \quad x, y, z \geq 0. \tag{1}$$

This follows since $f(u+v)$, $u \geq 0$, $v \geq 0$ is totally positive of order 2 (cf. [8, pp. 695–696]), and therefore

$$f\begin{pmatrix} 0, & y \\ x, & x+z \end{pmatrix} = \begin{vmatrix} f(x) & f(x+z) \\ f(x+y) & f(x+y+z) \end{vmatrix} \geq 0.$$

Now we are in a position to define the DFR and log-convexity properties for general random variables (see Barlow and Proschan [19]).

**Definition 2.4.** Let $X$ be a non-negative random variable with $G$ and $\overline{G} := 1 - G$ the corresponding distribution and reliability functions. $X$ (or $G$) is said to be: