Applied Mathematical Modelling 40 (2016) 5283–5291

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

# Anomalous spreading and misidentification of spatial random walk models☆

Reiichiro Kawai*

*School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia*

ARTICLE INFO

*Article history:*
Received 2 July 2014
Revised 30 October 2015
Accepted 14 December 2015
Available online 30 December 2015

*Keywords:*
Correlated random walk
Fractional Brownian motion
Long-range dependence
Mean-squared displacement
Turning angles
Super-diffusion

ABSTRACT

Anomalous diffusive spreading behaviors are often modeled with continuous-time random walks or fractional Brownian motion, while spatial directional preference is the key attribute of correlated random walks. In this paper, we construct and analyze a spatial random walk model with particular attention to the synthesis of anomalous diffusive behaviors and spatial directional preference, as well as statistical inference under discrete sampling. Anisotropic anomalous diffusive behaviors of the proposed model emerge from the long memory among step lengths, standing in contrast to little tangling diffusive trajectories of correlated random walk models, designed directly through the relative turning angles. With the help of the proposed model, we demonstrate difficulty in model identification among spatial random walk models and suggest that the identification procedure be planned and carried out with a great deal of caution.

© 2015 Elsevier Inc. All rights reserved.

## 1. Introduction

A random walk is a mathematical model consisting of a succession of random steps and is quite often employed to simplify a continuous path. In front-end practice, one is technically unable to monitor trajectories in a fully continuous manner, so a continuous spatial path is observed at discrete time points and the resulting positions at certain observation times are modeled with a suitable random walk model. There are numerous statistical properties of continuous spatial trajectories that a random walk model is desired to capture in macroscopic and microscopic manners. Those include finite or infinite variance, step lengths with light- or heavy-tailed distribution, either persistence, anti-persistence or no correlation in direction and in step length, etc. A variety of random walk models have been introduced and studied in many different fields with a view towards such statistical properties. Those models include correlated random walks [1,2], Lévy walks and flights [3–5], continuous-time random walks [6–9], and many other variations. In addition, based upon those models, the issue of interpretation of the given data has been widely investigated in, for instance, [3,10–15].

Among a variety of desired statistical properties of spatial trajectories, the directional preference and anomalous diffusive spreading behaviors have long attracted a great deal of interest, for instance, [16,17]. To capture the directional preference, the class of correlated random walk models has often been chosen as the first candidate. Correlated random walks consist of two major components in their modeling; one is a series of steps with either fixed or random lengths, while the other is a series of randomly distributed and correlated turning angles, between successive step vectors. Correlated random walk

☆ The author would like to thank two anonymous referees for valuable suggestions.

* Tel.: +61 2 9351 3376; fax: +61 2 9351 4534.
*E-mail address:* reiichiro.kawai@sydney.edu.au

http://dx.doi.org/10.1016/j.apm.2015.12.028

S0307-904X(15)00840-9/© 2015 Elsevier Inc. All rights reserved.

models can naturally exhibit the desired spatial directional preference directly by assigning an appropriate probability distribution, usually peaked around the zero angle, to the relative turning angles. As we will show rigorously later, correlated random walk models are not capable of producing non-diffusive spreading.

In the natural world, the mean-squared displacement of the observed spatial trajectories does not necessarily display linear growth in time. Instead, it is often sub-linear, for instance, in amorphous solids [9] and polymeric systems [18], and exhibits super-linear growth in turbulent media [19]. Anomalous diffusive behaviors are often modeled with fractional Brownian motion or continuous-time random walks mainly in one-dimensional setting. On the one hand, two-dimensional fractional Brownian motions studied in the literature (for instance, [4,20,21]) have spatially isotropic trajectories, that is, no directional preference can be exhibited. On the other hand, the class of continuous-time random walks have been investigated quite intensively for its capability of realizing a variety of diffusive modes through heavy-tailed distributions and correlations within and between waiting times and jump lengths. When making a statistical inference for trajectories based upon their discrete-time observations, the randomness of waiting times makes the law of increments quite involved in general. It is often difficult to go beyond checking whether the empirical mean-squared displacement evolves as expected relative to its theoretical behavior when the observation window increases.

In this paper, we construct and analyze a spatial discrete-time random walk model that exhibits both anomalous diffusive behaviors and spatial directional preference in a relatively simple manner. The main feature of the proposed model is to make use of the memory in step length with the help of fractional dependence among increments to generate anomalies, rather than incorporating directional memory into the relative turning angles as in correlated random walks. The proposed model is kept within the realm of discrete-time random walks so as to ease a statistical inference under realistic discrete sampling of trajectories. The proposed model exhibits different modes of the mean-squared displacement around the average trajectory, either asymptotically diffusive or super-diffusive, depending on the characterizing parameters. The property of different spatial diffusive modes may be quite counterintuitive, unlike in one dimensional modeling. The proposed model also serves as a concrete example of quite likely misidentification among spatial random walk models. In a similar manner to the well known difficulty in distinguishing between correlated random walks and Lévy flights [11,13,15], we demonstrate that, especially in situations where statistics are scarce, the proposed model can be confused easily with uncorrelated and correlated random walks.

## 2. Correlated random walk

As a direct counterpart of the proposed model, let us briefly review a class of correlated spatial random walk models. Consider a random walk in two dimension, whose location after $n$ steps is defined by,

$$\begin{bmatrix} Z_{n,x} \\ Z_{n,y} \end{bmatrix} := \sum_{k=1}^{n} \mathbb{V}_k := \sum_{k=1}^{n} Z_k \begin{bmatrix} \cos(\xi_k) \\ \sin(\xi_k) \end{bmatrix}. \tag{2.1}$$

Here, $\{\mathbb{V}_k\}_{k\in\mathbb{N}}$ indicate step vectors, $\{Z_k\}_{k\in\mathbb{N}}$ are nonnegative step lengths, while $\{\xi_k\}_{k\in\mathbb{N}}$ are absolute turning angles. Throughout the paper, we assume that $\{Z_k\}_{k\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ are independent and that the step lengths $\{Z_k\}_{k\in\mathbb{N}}$ are independent and identically distributed (iid) nonnegative random variables. In order to introduce the correlation among steps to the framework of the simple spatial random walk model, the absolute turning angles $\{\xi_k\}_{k\in\mathbb{N}}$ is formulated as a successive sum of relative turning angles $\{\eta_k\}_{k\in\mathbb{N}}$, which are iid random variables in $(-\pi, +\pi)$. Clearly, unless the relative turning angles are iid $U(-\pi, +\pi)$, the absolute turning angles $\{\xi_k\}_{k\in\mathbb{N}}$ are correlated among themselves and the resulting random walk model is thus non-Markovian. With the relative turning angles $\{\eta_k\}_{k\in\mathbb{N}}$ being iid $U(-\pi, +\pi)$, the correlated random walk reduces to a uncorrelated random walk and is then Markovian.

Without assuming a particular distribution of the relative turning angles $\{\eta_k\}_{k\in\mathbb{N}}$, Kareiva and Shigesada [1] derived the asymptotic behavior of the mean-squared displacement,

$$\mathbb{E}\left[Z_{n,x}^2 + Z_{n,y}^2\right] \sim n\left(\mathbb{E}\left[Z_1^2\right] + \frac{\mathbb{E}[\cos(\xi_1)]}{1 - \mathbb{E}[\cos(\xi_1)]}(\mathbb{E}[Z_1])^2\right), \tag{2.2}$$

where the asymptotic equivalence holds as $n \nearrow +\infty$. This indicates that the correlated random walk spreads only in a diffusive manner. We can also show that for $k_1 > k_2$,

$$\left|\frac{\mathbb{E}[\langle \mathbb{V}_{k_1}, \mathbb{V}_{k_2}\rangle]}{\sqrt{\mathrm{Var}(\|\mathbb{V}_{k_1}\|)\mathrm{Var}(\|\mathbb{V}_{k_2}\|)}}\right| \le \frac{(\mathbb{E}[Z_1])^2}{\mathrm{Var}(Z_1)} e^{-\frac{k_1-k_2}{2}(\ln a_1^{-1})} \searrow 0, \quad \text{as } k_1 - k_2 \nearrow +\infty, \tag{2.3}$$

where $a_1 := \mathbb{E}[\cos(\eta_1)]^2 + \mathbb{E}[\sin(\eta_1)]^2$. (For completeness sake, we provide the derivation of (2.3) in Appendix A in a concise manner.) Importantly, the constant $a_1$ is non-negative and strictly less than one, regardless of the distribution of the angle $\eta_1$. For instance, if $\{\eta_k\}_{k\geq 1}$ are iid centered Gaussian $\mathcal{N}(0, \lambda^2)$ for some $\lambda > 0$, then $a_1 = e^{-\lambda^2}$. Therefore, there may exist some directional dependence among steps, but still decaying exponentially fast. In the case of uncorrelated random walks, the mean-squared displacement (2.2) is explicit $\mathbb{E}[Z_{n,x}^2 + Z_{n,y}^2] = n\mathbb{E}[Z_1^2]$, while the directional correlation (2.3) vanishes.

To quantify the degree of trajectory tangling, we estimate the average number of steps a trajectory requires for one complete rotation, which we denote by $T_{\pm 2\pi}$ indicating the expected first passage time of the series $\sum_{k=1}^{n} \eta_k$ to the boundaries either $+2\pi$ or $-2\pi$. Clearly, a large first passage time implies little tangling. As mentioned earlier, the setting of

Download English Version:

https://daneshyari.com/en/article/1702954

Download Persian Version:

https://daneshyari.com/article/1702954

Daneshyari.com