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Perturbation-based simplified models for unsteady incompressible microchannel flows

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ABSTRACT

A fast simplified model for a pressure-driven, unsteady and incompressible flow in two dimensional microchannels with a variable cross section is presented. The model relies on a perturbation analysis of the equations of mass and momentum under the assumptions that the channels have a small aspect ratio of height to length and that they have a slowly varying cross-section. A key advantage of this model is that it provides (quasi-) analytic expressions for the velocity, pressure distribution and mass flow rate in terms of the pressure drop along the channel, which makes the evaluation of such expressions very fast, compared to simulating the full Navier–Stokes equations. This is especially important in applications where the flow rate-pressure drop relation can be cast into a circuit, to be potentially integrated in a more complex system. The model turns out to be fairly accurate for a wide range of non-negligible Reynolds and Strouhal numbers and a fairly large class of channels. Furthermore, the model offers a speed factor of at least 500 over classical CFD simulations.

1. Introduction

The precise control of liquid flows in microchannels plays a critical role in successfully designing and operating microfluidic devices, often called “lab-on-a-chip”. The microchannels in these devices seldom form an entity on their own but are rather integrated in a larger system; in fact, integrated microfluidic systems are the basis for many micro devices such as the μTAS, [1,2], PCR devices, [3], and digital micro-fluidics arrays [4].

Channels with dimensions ranging from nano to millimeters (microchannels) constitute the main building blocks of these micro devices, offering the transport of small volumes of fluid. For liquid flows, the flow length scale is much larger than the molecular scale, allowing for the continuum hypothesis to still hold, as well as for the no-slip condition to be valid at the walls [5]. Although microflows are generally laminar and nearly inertia free, the role of inertia is in many cases important, either due to unsteady effects or variations in the channel's cross-section. This is true for example for liquid flows oscillating at frequencies ranging from 1 Hz to 10 kHz, where the ranges of flow rates are in 1–100 μl/min, flowing in long microchannels of heights in the range of 10 μm to 1000 μm. It is also true for steady laminar liquid flows, with moderately large values of Reynolds number, flowing in long micro-channels of slightly varying cross sections. Applications include pressure or electro-osmotically driven flows for sampling, dispensing, and mixing [6,7].

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Nomenclature

x	coordinate in flow direction
z	coordinate along channel height
t	time
h_0	average channel height
L_x	channel length
α_z	channel aspect ratio, h_0/L_x
ϵ	perturbation parameter ($\epsilon = \alpha_z$)
γ	normalized channel height perturbation $\delta h/h_0$
β	channel slope
γ_0	sup γ
β_0	sup β
f	characteristic forcing frequency of the flow
ΔP_0	reference pressure drop along the channel
U	reference velocity in flow direction
W	reference velocity in transverse direction.
$\mathtt{Re}$	Reynolds number based on channel height
$\mathtt{St}$	Strouhal number based on channel height
$\mathtt{Po}$	Poiseuille number
u	velocity component in flow direction
w	velocity component in transverse direction
p	pressure
Q	volume flow rate
Δp_0	pressure drop along the channel

The level of complexity of these systems in terms of the number of components and the multi-physics they involve makes the task of simulating their behavior by numerically solving the systems of PDEs governing the conservation laws (Navier–Stokes equations) very costly; this is especially true because there is often a need to perform fast sweeps of several parameters in order to optimize the design of these structures. A faster simplified model for the fluid flow through such channels, that still captures the essential physical features of the device in question, is thus desirable.

In the proposed work, we present a very fast simplified model for a pressure-driven, unsteady and incompressible flow in two dimensional microchannels with a variable cross section. The model relies on a perturbation analysis of the equations of mass and momentum under the assumptions that: (i) the channels have a small aspect ratio of height to length, (ii) the channels have a variable cross-section, with a variation that is small in amplitude, but not necessarily slow. Long channels are typical in microflows; varying channels are likely to arise in applications involving compliant walls, where the channels' wall is also coupled to the flow [8]. This work can be viewed as necessary step to model some fluid-structure interaction problems in microsystems. The main advantage of our model is that it provides (quasi-) analytic expressions for the velocity, pressure distribution and mass flow rate in terms of the pressure drop along the channel, which makes the evaluation of such expressions very fast, compared to simulating the full Navier–Stokes equations. This is especially important in applications where the flow rate-pressure drop relation can be cast into a circuit, to be potentially integrated in a more complex system [9]. Another advantage of our model is that it turns out to be fairly accurate for a wide range of non-negligible Reynolds and Strouhal numbers.

The literature is abundant with models that account for steady incompressible flows in slowly varying channels, with little or no inertial effects. Some of the theoretical studies include the works of Manton [10], Wild et al. [11] and Ghosal [12], to cite a few. They range from rather simple channel wall models to more realistic complex ones. In [10] an asymptotic expansion is used to model fluid flows through axisymmetric tubes whose radius varies slowly in the axial direction. Analytic expressions for the pressure drop and the shear stress at the wall of the channel are obtained and the steady model is only valid for low Reynolds numbers. The author of [12] also uses an asymptotic expansion to analyze an electro-osmotic flow in microfluidic devices with no inertial effects. The method relies on the lubrication approximation, which is justified when the characteristic length scales for axial variation of the wall charge and cross-section are both large compared to a characteristic width of the channel. In [11], the authors conduct a theoretical and experimental study to analyze a steady flow in collapsible tubes, at low Reynolds numbers; the importance of the work is that it accounts for compliant channel walls, (where the cross-section is time variant and the model of the membrane is coupled to the flow). None of the aforementioned studies accounts for unsteady effects.

Unsteady effects on flows in varying microchannels are not very well studied in the literature, at least from the theoretical point of view. Some numerical studies include the work of Luo and Pedley [13], in which the authors numerically investigate the instability of the steady solution of the problem in [11]. The authors consider a flow in a parallel-sided channel with a segment of one wall replaced by a membrane under longitudinal tension; they show that the steady flow