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Analytical approximation and numerical studies of one-dimensional elliptic equation with random coefficients

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ABSTRACT

In this work, we study a one-dimensional elliptic equation with a random coefficient and derive an explicit analytical approximation. We model the random coefficient with a spatially varying random field, $K(x, \omega)$ with known covariance function. We derive the relation between the standard deviation of the solution $T(x, \omega)$ and the correlation length, η of $K(x, \omega)$. We observe that, the standard deviation, σ_T of the solution, $T(x, \omega)$, initially increases with the correlation length η up to a maximum value, $\sigma_{T,\max}$ at $\eta_{\max} \sim \sqrt{x(1-x)/3}$ and decreases beyond $\eta_{\max}$. We observe a scaling law between σ_T and η, that is, $\sigma_T \propto \eta^{1/2}$ for $\eta \to 0$ and $\sigma_T \propto \eta^{-1/2}$ for $\eta \to \infty$. We show that, for a small value of coefficient of variation ($\varepsilon_K = \sigma_K/\mu_K$) of the random coefficient, the solution $T(x, \omega)$ can be approximated with a Gaussian random field regardless of the underlying probability distribution of $K(x, \omega)$. This approximation is valid for large value of ε_K, if the correlation length, η of input random field $K(x, \omega)$ is small. We compare the analytical results with numerical ones obtained from Monte-Carlo method and polynomial chaos based stochastic collocation method. Under aforementioned conditions, we observe a good agreement between the numerical simulations and the analytical results. For a given random coefficient $K(x, \omega)$ with known mean and variance we can quickly estimate the variance of the solution at any location for a given correlation length. If the correlation length is not available which is the case in most practical situations, we can still use this analytical solution to estimate the maximum variance of the solution at any location.

1. Introduction

Elliptic equation is fundamental to many physical phenomena such as heat conduction, diffusion, mechanics, electrostatics, Darcy flow and many more. In this paper, we present analytical methods to study the solution of a one-dimensional elliptic equation with a random coefficient. We derive an explicit analytical solution for the stochastic elliptic equation and discuss the importance of the analytical solution and conditions under which this solution is valid. Often, in practice, a quick estimate of the variance of a solution is required to make preliminary decisions prior to a detailed computational analysis. An analytical study helps us to design and test algorithms for computational numerical studies.

In order to compare analytical solution with numerical solution, we solve the stochastic elliptic equation numerically using Monte-Carlo method, polynomial chaos based stochastic collocation method. There are many numerical methods such

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as Monte-Carlo methods [1], spectral stochastic Galerkin methods [2–8], stochastic collocation methods [9–14], perturbation methods to solve stochastic partial differential equations. Several numerical methods based on stochastic collocation and Galerkin methods [15–17] to solve elliptic equations with random coefficient. Finite element methods for structures with large variations is studied and applied to problems in solid mechanics in [18]. Mathematical models for random coefficients in a partial differential equations that satisfy physical and mathematical constraints are discussed in [19]. In [20] ANOVA based statistical methods are used to solve stochastic incompressible and compressible flows. A stochastic basis adaptation method is introduced in [21] to address the issue of curse of dimensionality by representing the quantities of interest (QoI) in a low dimensional stochastic space so that the computational cost is reduced significantly. In [22] analytical solution of fractal diffusion equations is studied. In [23], analytical solution of a one-dimensional advection-diffusion equation with several point sources was studied. Random Airy type differential equations and their numerical study based on power series solution is discussed in [24]. In [25], a Bayesian approach was used to recover the stochastic solution in both random and spatial domains.

Our contribution to this work is, to derive an analytical solution and study of the influence of correlation length, η on the standard deviation of the solution. We compare results from analytical study with those from numerical simulations. We show from both numerical and analytical results that the standard deviation of the solution increases as a function of the correlation length, η to a maximum at $\eta_{\max}$ and decreases beyond that point. σ_T follows square-root power law ($\sigma_T \propto \eta^{1/2}$) for $\eta \to 0$ and inverse square-root power law ($\sigma_T \propto \eta^{-1/2}$) for $\eta \to \infty$. Magnitude of $\sigma_{T,\max}$ varies for different spatial locations, x. As the magnitude of $\sigma_{T,\max}$ increases, the value $\eta_{\max}$ shifts towards right. We show that the solution can be approximated to a Gaussian random field regardless of the distribution of the random coefficient, $K(x, \omega)$ for small η. As we increase the value of ε_K and correlation length η, solution becomes non Gaussian and we observe larger error in analytical solution.

The paper is organized as follows. In Section 2, we introduce a steady state elliptic equation with a random coefficient and derive an analytical solution of stochastic elliptic equation and discuss assumptions under which this analytical solution holds. We discuss numerical methods namely, Monte-Carlo method, polynomial chaos based stochastic Galerkin and stochastic collocation methods in Section 3. Standard deviation and probability density function of the solution, obtained from numerical methods and analytical solution is compared in Section 4. Finally we conclude our paper with a discussion on the validity and limitations of the analytical solution and possible extension to the non-homogeneous steady state equation.

2. Analytical solution

Let us consider a one dimensional spatial domain, $D = [0, 1]$, and a complete probability space, (Ω, Σ, P) with sample space Ω, σ-algebra Σ and probability measure P. A stochastic elliptic partial differential equation can be written as: find a random field $T(x, \omega) : D \times \Omega \to \mathbb{R}$ such that the following equation holds P-almost surely (P-a.s.):

$$\frac{\partial}{\partial x}\left[K(x,\omega)\frac{\partial T(x,\omega)}{\partial x}\right] = 0, \quad x \in [0,1], \tag{1}$$
$$T(0,\omega) = T_0, T(1,\omega) = T_1,$$

where, x is the position and $T(x, \omega)$, is the solution field and we assume that the random coefficient, $K(x, \omega)$ is bounded stationary random field and strictly positive, that is,

$$0 < K_{\min} \le K(x,\omega) \le K_{\max} < \infty \qquad \text{in} \quad D \times \Omega, \tag{2}$$

where, $K(x, \omega)$ is a random field with known properties such as mean, covariance function and probability distribution. The steady-state solution $T(x, \omega)$, is also a stochastic field because of the randomness in the coefficient $K(x, \omega)$. Our objective is to find the statistical properties of $T(x, \omega)$, such as mean and standard deviation for a given random field $K(x, \omega)$.

The random coefficient $K(x, \omega)$, in the elliptic equation, Eq. (1) can be written as a series expansion that separates spatial variable, x and stochastic variable, ω. For this, second order statistics such as covariance function is required. A second order stationary process with known covariance function is used to model the random coefficient. We use following exponential covariance function for the input random field,

$$\mathrm{cov}_K(x_1, x_2) = \sigma_K^2 \exp\left(-\frac{|x_1 - x_2|}{\eta}\right), \tag{3}$$

where η is the correlation length of the input random field and the coefficient of variation, $\varepsilon_K = \sigma_K/\mu_K$, where μ_K and σ_K are the mean and standard deviation of $K(x, \omega)$. Random field $K(x, \omega)$ can be approximated using Karhunen–Loève expansion [26] as follows:

$$K(x,\omega) = K_0 + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\omega)K_i(x), \tag{4}$$